Reductions, Recursion and Divide and Conquer

Lecture 5
September 11, 2014
Part I

Reductions and Recursion
Reduction

Reducing problem $A$ to problem $B$:

1. Algorithm for $A$ uses algorithm for $B$ as a *black box*
Reduction

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Q: How do you hunt a blue elephant?
A: With a blue elephant gun.
Reduction

Reducing problem A to problem B:

1. Algorithm for A uses algorithm for B as a black box

Q: How do you hunt a blue elephant?
A: With a blue elephant gun.

Q: How do you hunt a red elephant?
A: Hold his trunk shut until he turns blue, and then shoot him with the blue elephant gun.
Reduction

Reducing problem A to problem B:

1. Algorithm for A uses algorithm for B as a black box

Q: How do you hunt a blue elephant?
A: With a blue elephant gun.

Q: How do you hunt a red elephant?
A: Hold his trunk shut until he turns blue, and then shoot him with the blue elephant gun.

Q: How do you shoot a white elephant?
A: Embarrass it till it becomes red. Now use your algorithm for hunting red elephants.
Problem: Given an array $A$ of $n$ integers, are there any duplicates in $A$?

Naive algorithm:

```plaintext
for i = 1 to n - 1 do
    for j = i + 1 to n do
        if (A[i] = A[j])
            return YES

return NO
```

Running time: $O(n^2)$
UNIQUENESS: Distinct Elements Problem

Problem Given an array $A$ of $n$ integers, are there any duplicates in $A$?

Naive algorithm:

```plaintext
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      return YES
  
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Running time: $O(n^2)$
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      return YES
  
return NO
```

Running time: $O(n^2)$
Problem Given an array $A$ of $n$ integers, are there any duplicates in $A$?

Naive algorithm:


def check_duplicates(A, n):
    for i in range(1, n):
        for j in range(i+1, n):
            if A[i] == A[j]:
                return True
    return False

Running time: $O(n^2)$
Reduction to Sorting

Sort $A$
for $i = 1$ to $n - 1$ do
    if ($A[i] = A[i + 1]$) then
        return YES
    return NO

Running time: $O(n)$ plus time to sort an array of $n$ numbers

Important point: algorithm uses sorting as a black box
Reduction to Sorting

Sort $A$

for $i = 1$ to $n - 1$ do

  if ($A[i] = A[i + 1]$) then
    return YES

return NO

Running time: $O(n)$ plus time to sort an array of $n$ numbers

Important point: algorithm uses sorting as a black box
Two sides of Reductions

Suppose problem A reduces to problem B

1. Positive direction: Algorithm for B implies an algorithm for A
2. Negative direction: Suppose there is no “efficient” algorithm for A then it implies no efficient algorithm for B (technical condition for reduction time necessary for this)

Example: Distinct Elements reduces to Sorting in $O(n)$ time

1. An $O(n \log n)$ time algorithm for Sorting implies an $O(n \log n)$ time algorithm for Distinct Elements problem.
2. If there is no $o(n \log n)$ time algorithm for Distinct Elements problem then there is no $o(n \log n)$ time algorithm for Sorting.
Two sides of Reductions

Suppose problem A reduces to problem B

1. **Positive direction:** Algorithm for B implies an algorithm for A
2. **Negative direction:** Suppose there is no “efficient” algorithm for A then it implies no efficient algorithm for B (technical condition for reduction time necessary for this)

**Example:** Distinct Elements reduces to Sorting in \( O(n) \) time

1. An \( O(n \log n) \) time algorithm for Sorting implies an \( O(n \log n) \) time algorithm for Distinct Elements problem.
2. If there is *no* \( o(n \log n) \) time algorithm for Distinct Elements problem then there is *no* \( o(n \log n) \) time algorithm for Sorting.
If we can reduce from A to B, and from B to A then...

(A) The fastest running times to solve A or B are the same.

(B) A and B might still require different running times to be solved efficiently.

(C) Then A and B are the same problem, and this can not happen.
Recursion

**Reduction:** reduce one problem to another

**Recursion:** a special case of reduction

1. reduce problem to a *smaller* instance of *itself*
2. self-reduction

1. Problem instance of size $n$ is reduced to *one or more* instances of size $n - 1$ or less.
2. For termination, problem instances of small size are solved by some other method as *base cases*
Recursion

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Recursion

1. Recursion is a very powerful and fundamental technique
2. Basis for several other methods
   1. Divide and conquer
   2. Dynamic programming
   3. Enumeration and branch and bound etc
   4. Some classes of greedy algorithms
3. Makes proof of correctness easy (via induction)
4. Recurrences arise in analysis
Selection Sort

Sort a given array $A[1..n]$ of integers.

Recursive version of Selection sort.

```
SelectSort(A[1..n]) :
    if n = 1 return
    Find smallest number in A. Let A[i] be smallest number
    Swap A[1] and A[i]
    SelectSort(A[2..n])
```

$T(n)$: time for SelectSort on an $n$ element array.

$T(n) = T(n - 1) + n$ for $n > 1$ and $T(1) = 1$ for $n = 1$

$T(n) = \Theta(n^2)$. 
Selection Sort

Sort a given array \( A[1..n] \) of integers.

Recursive version of Selection sort.

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\text{SelectSort}(A[1..n]):
\begin{align*}
\text{if } n = 1 & \text{ return } \\
\text{Find smallest number in } A. \text{ Let } A[i] \text{ be smallest number} \\
\text{Swap } A[1] \text{ and } A[i] \\
\text{SelectSort}(A[2..n])
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$T(n) = \Theta(n^2)$. 
Move stack of $n$ disks from peg 0 to peg 2, one disk at a time. 
**Rule:** cannot put a larger disk on a smaller disk. 
**Question:** what is a strategy and how many moves does it take?
The Tower of Hanoi algorithm; ignore everything but the bottom disk
Recursive Algorithm

\[ \text{Hanoi}(n, \text{src}, \text{dest}, \text{tmp}): \]
\[ \text{if } (n > 0) \text{ then} \]
\[ \text{Hanoi}(n - 1, \text{src}, \text{tmp}, \text{dest}) \]
\[ \text{Move disk } n \text{ from src to dest} \]
\[ \text{Hanoi}(n - 1, \text{tmp}, \text{dest}, \text{src}) \]

\[ T(n) : \text{ time to move } n \text{ disks via recursive strategy} \]

\[ T(n) = 2T(n - 1) + 1 \quad n > 1 \quad \text{and } T(1) = 1 \]
Recursive Algorithm

\textbf{Hanoi}(n, \text{src}, \text{dest}, \text{tmp}): \\
\textbf{if} (n > 0) \textbf{then} \\
\hspace{1cm} \textbf{Hanoi}(n - 1, \text{src}, \text{tmp}, \text{dest}) \\
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  &\text{if } (n > 0) \text{ then} \\
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\textbf{T(n)}: \text{ time to move } n \text{ disks via recursive strategy}

\[ T(n) = 2T(n - 1) + 1 \quad n > 1 \quad \text{and } T(1) = 1 \]
Analysis

\[
T(n) = 2T(n - 1) + 1 \\
= 2^2T(n - 2) + 2 + 1 \\
= \ldots \\
= 2^iT(n - i) + 2^{i-1} + 2^{i-2} + \ldots + 1 \\
= \ldots \\
= 2^{n-1}T(1) + 2^{n-2} + \ldots + 1 \\
= 2^{n-1} + 2^{n-2} + \ldots + 1 \\
= (2^n - 1)/(2 - 1) = 2^n - 1
\]
Non-Recursive Algorithms for Tower of Hanoi

Pegs numbered 0, 1, 2

Non-recursive Algorithm 1:
1. Always move smallest disk forward if \( n \) is even, backward if \( n \) is odd.
2. Never move the same disk twice in a row.
3. Done when no legal move.

Non-recursive Algorithm 2:
1. Let \( \rho(n) \) be the smallest integer \( k \) such that \( n/2^k \) is not an integer. Example: \( \rho(40) = 4, \rho(18) = 2 \).
2. In step \( i \) move disk \( \rho(i) \) forward if \( n - i \) is even and backward if \( n - i \) is odd.

Moves are exactly same as those of recursive algorithm. Prove by induction.
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Divide and Conquer Paradigm

Divide and Conquer is a common and useful type of recursion

### Approach

1. Break problem instance into smaller instances - divide step
2. **Recursively** solve problem on smaller instances
3. Combine solutions to smaller instances to obtain a solution to the original instance - conquer step

**Question:** Why is this not plain recursion?

1. In divide and conquer, each smaller instance is typically at least a constant factor smaller than the original instance which leads to efficient running times.
2. There are many examples of this particular type of recursion that it deserves its own treatment.
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Input  Given an array of $n$ elements
Goal  Rearrange them in ascending order
Input: Array $A[1 \ldots n]$
Merge Sort [von Neumann]

1. **Input:** Array $A[1 \ldots n]$

2. Divide into subarrays $A[1 \ldots m]$ and $A[m + 1 \ldots n]$, where $m = \lfloor n/2 \rfloor$
Merge Sort [von Neumann]

1. **Input:** Array $A[1 \ldots n]$ 

2. **Divide into subarrays** $A[1 \ldots m]$ and $A[m + 1 \ldots n]$, where $m = \lfloor n/2 \rfloor$

3. **Recursively** MergeSort $A[1 \ldots m]$ and $A[m + 1 \ldots n]$
Merge Sort [von Neumann]

MergeSort

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4. Merge the sorted arrays
Merge Sort [von Neumann]

MergeSort

1. **Input**: Array $A[1 \ldots n]$

2. Divide into subarrays $A[1 \ldots m]$ and $A[m + 1 \ldots n]$, where $m = \lfloor n/2 \rfloor$

3. Recursively **MergeSort** $A[1 \ldots m]$ and $A[m + 1 \ldots n]$

4. Merge the sorted arrays
Merging Sorted Arrays

1. Use a new array $C$ to store the merged array
2. Scan $A$ and $B$ from left-to-right, storing elements in $C$ in order

A G L O R H I M S T
A G H I L M O R S T
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\[
\begin{align*}
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\text{A G H I L M O R S T}
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Merging Sorted Arrays

1. Use a new array $C$ to store the merged array
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\[
\begin{array}{cccccccccccc}
A & G & L & O & R \\
H & I & M & S & T \\
A & G & H & I & L & M & O & R & S & T
\end{array}
\]

3. Merge two arrays using only constantly more extra space (in-place merge sort): doable but complicated and typically impractical.
**Running Time**

\( T(n) \): time for merge sort to sort an \( n \) element array

\[
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn
\]

What do we want as a solution to the recurrence?

Almost always only an *asymptotically* tight bound. That is we want to know \( f(n) \) such that \( T(n) = \Theta(f(n)) \).

1. \( T(n) = O(f(n)) \) - upper bound
2. \( T(n) = \Omega(f(n)) \) - lower bound
### Running Time

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Solving Recurrences: Some Techniques

1. Know some basic math: geometric series, logarithms, exponentials, elementary calculus
2. Expand the recurrence and spot a pattern and use simple math
3. Recursion tree method — imagine the computation as a tree
4. Guess and verify — useful for proving upper and lower bounds even if not tight bounds

Albert Einstein: “Everything should be made as simple as possible, but not simpler.”

Know where to be loose in analysis and where to be tight. Comes with practice, practice, practice!
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Unroll the recurrence. $T(n) = 2T(n/2) + cn$
Recursion Trees

**MergeSort**: $n$ is a power of 2

1. **Unroll the recurrence.** $T(n) = 2T(n/2) + cn$

2. **Identify a pattern.** At the $i$th level total work is $cn$. 
Recursion Trees

MergeSort: \( n \) is a power of 2

1. Unroll the recurrence. \( T(n) = 2T(n/2) + cn \)

2. Identify a pattern. At the \( i \)th level total work is \( cn \).
Unroll the recurrence. $T(n) = 2T(n/2) + cn$

Identify a pattern. At the $i$th level total work is $cn$.

Sum over all levels. The number of levels is $\log n$. So total is $cn \log n = \Theta(n \log n)$. 

Recursion Trees

MergeSort: $n$ is a power of 2
Unroll the recurrence. \( T(n) = 2T(n/2) + cn \)

Identify a pattern. At the \( i \)th level total work is \( cn \).

Sum over all levels. The number of levels is \( \log n \). So total is \( cn \log n = O(n \log n) \).
Recursion Trees
An illustrated example...
Recursion Trees

An illustrated example...

Work in each node

n
\( cn \)

\( n/2 \)
\( cn/2 \)
\( n/2 \)
\( cn/2 \)

\( n/4 \)
\( cn/4 \)
\( n/4 \)
\( cn/4 \)
\( n/4 \)
\( cn/4 \)

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Recursion Trees

An illustrated example...

![Recursion Tree Diagram]

- Work in each node

$n$ $cn$
$n/2$ $cn/2$
$n/4$ $cn/4$
$n/8$ $cn/8$

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Recursion Trees
An illustrated example...

\[
\log n \begin{cases} 
  cn = cn \\
  \frac{cn}{2} + \frac{cn}{2} = cn \\
  \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} = cn \\
  \vdots \\
  = cn
\end{cases}
\]
Recursion Trees

An illustrated example...

\[ \log n \left\{ \begin{array}{c}
\frac{cn}{2} + \frac{cn}{4} + \frac{cn}{4} + \ldots + \frac{cn}{4} = cn \\
\frac{cn}{2} + \frac{cn}{4} + \frac{cn}{4} + \ldots = cn \\
\frac{cn}{2} + \frac{cn}{4} + \frac{cn}{4} + \ldots = cn \\
\frac{cn}{2} + \frac{cn}{4} + \frac{cn}{4} + \ldots = cn \\
\end{array} \right. \]

\[ = cn \log n = O(n \log n) \]
When \( n \) is not a power of \( 2 \), the running time of \texttt{MergeSort} is expressed as

\[
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn
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When \( n \) is not a power of 2, the running time of MergeSort is expressed as

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1. When \( n \) is not a power of 2, the running time of MergeSort is expressed as

2. \( n_1 = 2^{k-1} < n \leq 2^k = n_2 \) (\( n_1, n_2 \) powers of 2).
1. When \( n \) is not a power of 2, the running time of \textbf{MergeSort} is expressed as

\[
T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + T\left(\left\lceil \frac{n}{2} \right\rceil \right) + cn
\]

2. \( n_1 = 2^{k-1} < n \leq 2^k = n_2 \) (\( n_1, n_2 \) powers of 2).

3. \( T(n_1) < T(n) \leq T(n_2) \) (Why?).
When $n$ is not a power of 2, the running time of MergeSort is expressed as

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

1. $n_1 = 2^{k-1} < n \leq 2^k = n_2$ ($n_1, n_2$ powers of 2).
2. $T(n_1) < T(n) \leq T(n_2)$ (Why?).
3. $T(n) = \Theta(n \log n)$ since $n/2 \leq n_1 < n \leq n_2 \leq 2n$. 

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Recursion Trees

**MergeSort**: \( n \) is not a power of 2
$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

Observation: For any number $x$, $\lfloor x/2 \rfloor + \lceil x/2 \rceil = x$. 
MergeSort Analysis
When $n$ is not a power of 2: Guess and Verify

If $n$ is power of 2 we saw that $T(n) = \Theta(n \log n)$. Can guess that $T(n) = \Theta(n \log n)$ for all $n$. Verify? proof by induction!

Induction Hypothesis: $T(n) \leq 2cn \log n$ for all $n \geq 1$
Base Case: $n = 1$. $T(1) = 0$ since no need to do any work and $2cn \log n = 0$ for $n = 1$.
Induction Step Assume $T(k) \leq 2ck \log k$ for all $k < n$ and prove it for $k = n$. 

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If $n$ is power of 2 we saw that $T(n) = \Theta(n \log n)$. Can guess that $T(n) = \Theta(n \log n)$ for all $n$. Verify? proof by induction!

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**Base Case:** $n = 1$. $T(1) = 0$ since no need to do any work and $2cn \log n = 0$ for $n = 1$.

**Induction Step** Assume $T(k) \leq 2ck \log k$ for all $k < n$ and prove it for $k = n$. 
Induction Step

We have

\[
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn
\]

\[
\leq 2c \lfloor n/2 \rfloor \log \lfloor n/2 \rfloor + 2c \lceil n/2 \rceil \log \lceil n/2 \rceil + cn \quad \text{(by induction)}
\]

\[
\leq 2c \lfloor n/2 \rfloor \log \lfloor n/2 \rfloor + 2c \lceil n/2 \rceil \log \lceil n/2 \rceil + cn
\]

\[
\leq 2c (\lfloor n/2 \rfloor + \lceil n/2 \rceil) \log \lfloor n/2 \rfloor + cn
\]

\[
\leq 2cn \log \lceil n/2 \rceil + cn
\]

\[
\leq 2cn \log (2n/3) + cn \quad \text{(since } \lfloor n/2 \rfloor \leq 2n/3 \text{ for all } n \geq 2)\]

\[
\leq 2cn \log n + cn(1 - 2 \log 3/2)
\]

\[
\leq 2cn \log n + cn(\log 2 - \log 9/4)
\]

\[
\leq 2cn \log n
\]
Guess and Verify

The math worked out like magic!
Why was $2cn \log n$ chosen instead of say $4cn \log n$?

1. Do not know upfront what constant to choose.
2. Instead assume that $T(n) \leq \alpha cn \log n$ for some constant $\alpha$. $\alpha$ will be fixed later.
3. Need to prove that for $\alpha$ large enough the algebra succeeds.
4. In our case... need $\alpha$ such that $\alpha \log 3/2 > 1$.
5. Typically, do the algebra with $\alpha$ and then show that it works... ... if $\alpha$ is chosen to be sufficiently large constant.

How do we know which function to guess?
We don’t so we try several “reasonable” functions. With practice and experience we get better at guessing the right function.
Guess and Verify

The math worked out like magic!
Why was $2cn \log n$ chosen instead of say $4cn \log n$?

1. Do not know upfront what constant to choose.
2. Instead assume that $T(n) \leq \alpha cn \log n$ for some constant $\alpha$. $\alpha$ will be fixed later.
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How do we know which function to guess?
We don’t so we try several “reasonable” functions. With practice and experience we get better at guessing the right function.
Guess and Verify
What happens if the guess is wrong?

1. Guessed that the solution to the **MergeSort** recurrence is $T(n) = O(n)$.

2. Try to prove by induction that $T(n) \leq \alpha cn$ for some constant $\alpha$. **Induction Step:** attempt

   \[
   T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn \\
   \leq \alpha c \lfloor n/2 \rfloor + \alpha c \lceil n/2 \rceil + cn \\
   \leq \alpha cn + cn \\
   \leq (\alpha + 1)cn
   \]

   But need to show that $T(n) \leq \alpha cn$!

3. So guess does not work for *any* constant $\alpha$. Suggests that our guess is incorrect.
Selection Sort vs Merge Sort

1. Selection Sort spends $O(n)$ work to reduce problem from $n$ to $n - 1$ leading to $O(n^2)$ running time.

2. Merge Sort spends $O(n)$ time after reducing problem to two instances of size $n/2$ each. Running time is $O(n \log n)$

**Question:** Merge Sort splits into 2 (roughly) equal sized arrays. Can we do better by splitting into more than 2 arrays? Say $k$ arrays of size $n/k$ each?
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Quick Sort

Quick Sort [Hoare]

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself. Linear scan of array does it. Time is $O(n)$
3. Recursively sort the subarrays, and concatenate them.

Example:

1. array: 16, 12, 14, 20, 5, 3, 18, 19, 1
2. pivot: 16
3. split into 12, 14, 5, 3, 1 and 20, 19, 18 and recursively sort
4. put them together with pivot in middle
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Let $k$ be the rank of the chosen pivot. Then,

$$T(n) = T(k - 1) + T(n - k) + O(n)$$
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\[
T(n) = T(k - 1) + T(n - k) + O(n)
\]

If \( k = \lceil n/2 \rceil \) then
\[
T(n) = T(\lceil n/2 \rceil - 1) + T(\lfloor n/2 \rfloor) + O(n) \leq 2T(n/2) + O(n).
\]

Then, \( T(n) = O(n \log n) \).
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Theoretically, median can be found in linear time.
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Theoretically, median can be found in linear time.

Typically, pivot is the first or last element of array. Then,
$$T(n) = \max_{1 \leq k \leq n} (T(k - 1) + T(n - k) + O(n))$$

In the worst case $T(n) = T(n - 1) + O(n)$, which means $T(n) = O(n^2)$. Happens if array is already sorted and pivot is always first element.
Part III

Fast Multiplication
Problem  Given two \( n \)-digit numbers \( x \) and \( y \), compute their product.

Grade School Multiplication

Compute “partial product” by multiplying each digit of \( y \) with \( x \) and adding the partial products.

\[
\begin{array}{c}
3141 \\
\times 2718 \\
\hline
25128 \\
3141 \\
21987 \\
6282 \\
\hline
8537238
\end{array}
\]
Time Analysis of Grade School Multiplication

1. Each partial product: $\Theta(n)$
2. Number of partial products: $\Theta(n)$
3. Addition of partial products: $\Theta(n^2)$
4. Total time: $\Theta(n^2)$
A Trick of Gauss

Carl Fridrich Gauss: 1777–1855 “Prince of Mathematicians”

Observation: Multiply two complex numbers: \((a + bi)\) and \((c + di)\)

\[(a + bi)(c + di) = ac - bd + (ad + bc)i\]

How many multiplications do we need?

Only 3! If we do extra additions and subtractions.
Compute \(ac, bd, (a + b)(c + d)\). Then
\[(ad + bc) = (a + b)(c + d) - ac - bd\]
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\[(ad + bc) = (a + b)(c + d) - ac - bd\]
Divide and Conquer

Assume \( n \) is a power of \( 2 \) for simplicity and numbers are in decimal.

1. \( x = x_{n-1}x_{n-2} \cdots x_0 \) and \( y = y_{n-1}y_{n-2} \cdots y_0 \)

2. \( x = 10^{n/2}x_L + x_R \) where \( x_L = x_{n-1} \cdots x_{n/2} \) and \( x_R = x_{n/2-1} \cdots x_0 \)

3. \( y = 10^{n/2}y_L + y_R \) where \( y_L = y_{n-1} \cdots y_{n/2} \) and \( y_R = y_{n/2-1} \cdots y_0 \)

Therefore

\[
x y = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R)
= 10^n x_L y_L + 10^{n/2} (x_L y_R + x_R y_L) + x_R y_R
\]
Example

\[1234 \times 5678 = (100 \times 12 + 34) \times (100 \times 56 + 78) = 10000 \times 12 \times 56 + 100 \times (12 \times 78 + 34 \times 56) + 34 \times 78\]
Time Analysis

\[ xy = (10^{n/2} x_L + x_R)(10^{n/2} y_L + y_R) \]
\[ = 10^n x_L y_L + 10^{n/2} (x_L y_R + x_R y_L) + x_R y_R \]

4 recursive multiplications of number of size \( n/2 \) each plus 4 additions and left shifts (adding enough 0’s to the right)

\[
T(n) = 4T(n/2) + O(n) \quad T(1) = O(1)
\]

\( T(n) = \Theta(n^2) \). No better than grade school multiplication!

Can we invoke Gauss’s trick here?
\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \]
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\[
xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R)
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Gauss trick: \( x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R \)

Recursively compute only \( x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R) \).

Time Analysis

Running time is given by

\[ T(n) = 3T(n/2) + O(n) \quad T(1) = O(1) \]

which means \( T(n) = O(n^{\log_2 3}) = O(n^{1.585}) \)
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State of the Art

Schönhage-Strassen 1971: $O(n \log n \log \log n)$ time using Fast-Fourier-Transform (FFT)

Martin Führer 2007: $O(n \log n 2^{O(\log^* n)})$ time

Conjecture

There is an $O(n \log n)$ time algorithm.
Analyzing the Recurrences

1. Basic divide and conquer: \( T(n) = 4T(n/2) + O(n) \), \( T(1) = 1 \). **Claim:** \( T(n) = \Theta(n^2) \).

2. Saving a multiplication: \( T(n) = 3T(n/2) + O(n) \), \( T(1) = 1 \). **Claim:** \( T(n) = \Theta(n^{1+\log 1.5}) \)

Use recursion tree method:

1. In both cases, depth of recursion \( L = \log n \).

2. Work at depth \( i \) is \( 4^i n/2^i \) and \( 3^i n/2^i \) respectively: number of children at depth \( i \) times the work at each child.

3. Total work is therefore \( n \sum_{i=0}^{L} 2^i \) and \( n \sum_{i=0}^{L} (3/2)^i \) respectively.
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