This exam lasts 120 minutes.
Write your answers in the separate answer booklet.
Please return this question sheet and your cheat sheet with your answers.

1. Each of these ten questions has one of the following five answers:

A: $\Theta(1)$ B: $\Theta(\log n)$ C: $\Theta(n)$ D: $\Theta(n \log n)$ E: $\Theta(n^2)$

(a) What is $\dfrac{n^5 - 3n^3 - 5n + 4}{3n^4 - 2n^2 + n - 7}$?

(b) What is $\displaystyle\sum_{i=1}^{n} \frac{n}{i}$?

(c) What is $\displaystyle\sum_{i=1}^{n} \frac{i}{n}$?

(d) How many bits are required to write the integer 10^n in binary?

(e) What is the solution to the recurrence $E(n) = E(n/3) + E(n/4) + E(n/5) + n/6$?

(f) What is the solution to the recurrence $F(n) = 16\,F(n/4 + 2) + n$?

(g) What is the solution to the recurrence $G(n) = G(n/2) + 2G(n/4) + n$?

(h) The *total path length* of a binary tree is the sum of the depths of all nodes. What is the total path length of a perfectly balanced n-node binary tree?

(i) Consider the following recursive function, defined in terms of two fixed arrays $A[1\,..\,n]$ and $B[1\,..\,n]$:

$$WTF(i,j) = \begin{cases} 0 & \text{if } i > j \\ \max \left\{ \begin{array}{c} (A[i] - B[j])^2 + WTF(i+1, j-1) \\ A[i]^2 + WTF(i+1, j) \\ B[i]^2 + WTF(i, j-1) \end{array} \right\} & \text{otherwise} \end{cases}$$

How long does it take to compute $WTF(1,n)$ using dynamic programming?

(j) Voyager 1 recently became the first made-made object to reach interstellar space. Currently the spacecraft is about 18 billion kilometers (roughly 60,000 light seconds) from Earth, traveling outward at approximately 17 kilometers per second (approximately 1/18000 of the speed of light). Voyager carries a golden record containing over 100 digital images abd approximately one hour of sound recordings. In digital form, the recording would require about 1 gigabyte. Voyager can transmit data back to Earth at approximately 1400 bits per second. Suppose the engineers at JPL sent instructions to Voyager 1 to send the complete contents of the Golden Record back to Earth; how many seconds would they have to wait to receive the entire record?

2. Suppose we are given an array $A[0..n+1]$ with fencepost values $A[0] = A[n+1] = -\infty$. We say that an element $A[x]$ is a *local maximum* if it is less than or equal to its neighbors, or more formally, if $A[x-1] \leq A[x]$ and $A[x] \geq A[x+1]$. For example, there are five local maxima in the following array:

$-\infty$	6	7	2	1	3	7	5	4	9	9	3	4	8	6	$-\infty$

We can obviously find a local maximum in $O(n)$ time by scanning through the array. Describe and analyze an algorithm that returns the index of one local maximum in $O(\log n)$ time. *[Hint: With the given boundary conditions, the array **must** have at least one local maximum. Why?]*

3. ***Prove*** that in any binary tree, the number of nodes with no children (leaves) is exactly one more than the number of nodes with two children. (Remember that a binary tree can have nodes with only one child.)

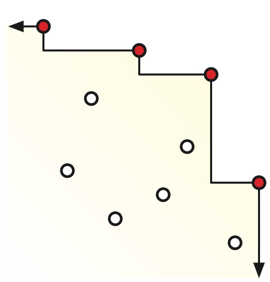
4. A string x is a *supersequence* of a string y if we can obtain x by inserting zero or more letters into y, or equivalently, if y is a subsequence of x. For example, the string **D**YN**AM**IC**P**ROG**RA**MMIN**G** is a supersequence of the string DAMPRAG.

A *palindrome* is any string that is exactly the same as its reversal, like I, DAD, HANNAH, AIBOHPHOBIA (fear of palindromes), or the empty string.

Describe and analyze an algorithm to find the length of the shortest supersequence of a given string that is also a palindrome.

For example, the 11-letter string E**HE**C**ADAC**E**HE** is the shortest palindrome supersequence of HEADACHE, so given the string HEADACHE as input, your algorithm should output the number 11.

5. Suppose you are given a set P of n points in the plane. A point $p \in P$ is *maximal* in P if no other point in P is both above and to the right of P. Intuitively, the maximal points define a “staircase” with all the other points of P below it.



A set of ten points, four of which are maximal.

Describe and analyze an algorithm to compute the number of maximal points in P in $O(n \log n)$ time. For example, given the ten points shown above, your algorithm should return the integer 4.