Breadth First Search, Dijkstra's Algorithm for Shortest Paths

Lecture 3

January 25, 2011

Part I

Breadth First Search

Breadth First Search (BFS)

Overview

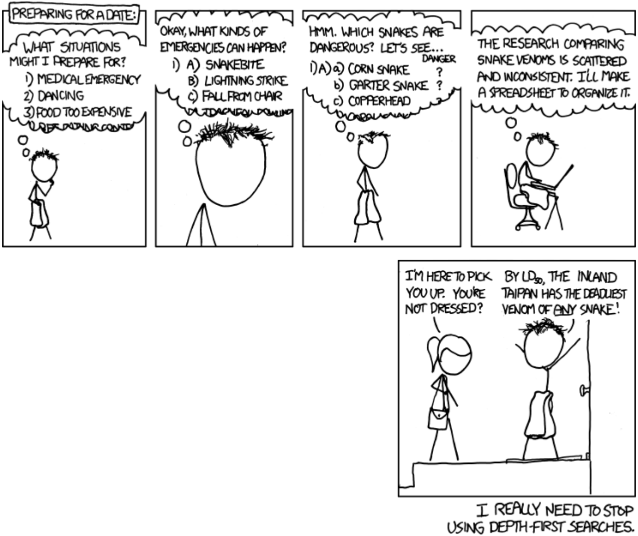
(A) **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex **s** (the start vertex).

As such...

- **DFS** good for exploring graph structure
- **BFS** good for exploring *distances*

xkcd take on BFS



Queue Data Structure

Queues

A **queue** is a list of elements which supports the operations:

- **enqueue**: Adds an element to the end of the list
- **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.

BFS Algorithm

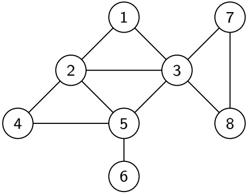
Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

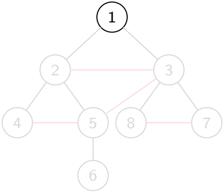
```
BFS(s)
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u, v) to T
                Mark v as visited and enq(v)
```

Proposition

$\mathsf{BFS}(s)$ *runs in* $O(n + m)$ *time.*

BFS: An Example in Undirected Graphs





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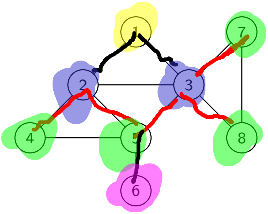
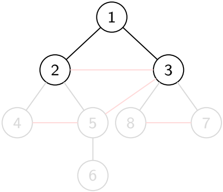
BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs





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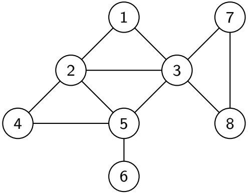

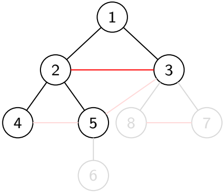

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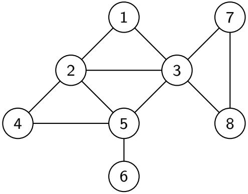

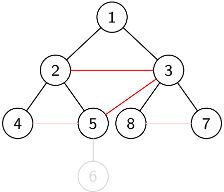

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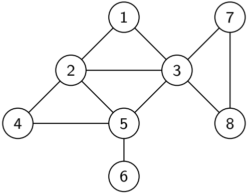

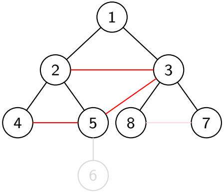

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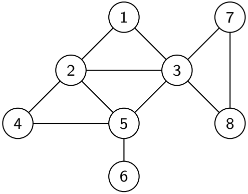

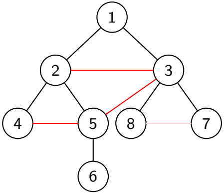

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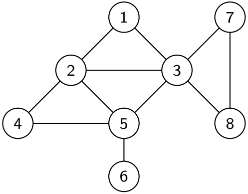

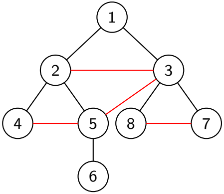

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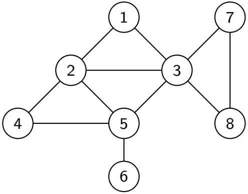

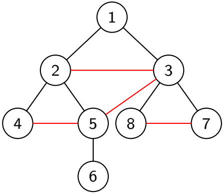

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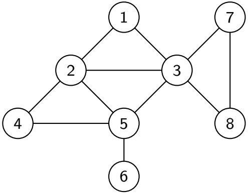

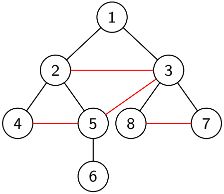

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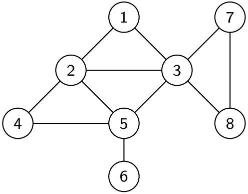

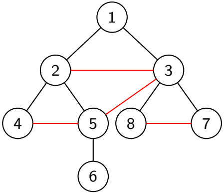

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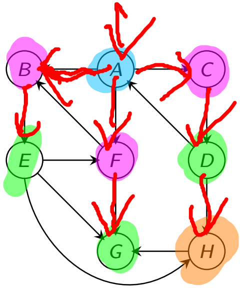
BFS: An Example in Directed Graphs



BFS with Distance

```
BFS(s)
    Mark all vertices as unvisited and for each v set dist(v) = ∞
    Initialize search tree T to be empty
    Mark vertex s as visited and set dist(s) = 0
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u) do
            if v is not visited do
                add edge (u, v) to T
                Mark v as visited, enq(v)
                and set dist(v) = dist(u) + 1
```

Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of **BFS**(**s**)

(A) *The search tree contains exactly the set of vertices in the connected component of* **s**.

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* **u** *is visited before* **v**.

(C) *For every vertex* **u**, $\mathrm{dist}(\mathbf{u})$ *is indeed the length of shortest path from* **s** *to* **u**.

(D) *If* $\mathbf{u}, \mathbf{v}$ *are in connected component of* **s** *and* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is an edge of* **G**, *then either* **e** *is an edge in the search tree, or* $|\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v})| \leq 1$.

Proof.

Exercise. □

Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of **BFS**$(\mathbf{s})$*:*

(A) *The search tree contains exactly the set of vertices reachable from* $\mathbf{s}$

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* $\mathbf{u}$ *is visited before* $\mathbf{v}$

(C) *For every vertex* $\mathbf{u}$, $\mathrm{dist}(\mathbf{u})$ *is indeed the length of shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$

(D) *If* $\mathbf{u}$ *is reachable from* $\mathbf{s}$ *and* $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ *is an edge of* $\mathbf{G}$, *then either* $\mathbf{e}$ *is an edge in the search tree, or* $\mathrm{dist}(\mathbf{v}) - \mathrm{dist}(\mathbf{u}) \leq 1$. *Not necessarily the case that* $\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v}) \leq 1$.

Proof.

Exercise. □

BFS with Layers

```
BFSLayers(s):
Mark all vertices as unvisited and initialize T to be empty
Mark s as visited and set L_0 = {s}
i = 0
while L_i is not empty do
        initialize L_{i+1} to be an empty list
        for each u in L_i do
            for each edge (u, v) ∈ Adj(u) do
            if v is not visited
                    mark v as visited
                    add (u, v) to tree T
                    add v to L_{i+1}
        i = i + 1
```

Running time: $O(n + m)$

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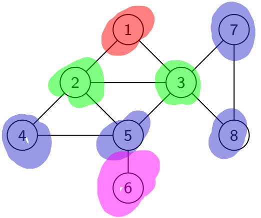
Example



BFS with Layers: Properties

Proposition

The following properties hold on termination of **BFSLayers**$(\mathbf{s})$.

- **BFSLayers**$(\mathbf{s})$ *outputs a* **BFS** *tree*
- $\mathbf{L_i}$ *is the set of vertices at distance exactly* $\mathbf{i}$ *from* $\mathbf{s}$
- *If* $\mathbf{G}$ *is undirected, each edge* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is one of three types:*
  - **tree** *edge between two* consecutive *layers*
  - *non-tree* **forward**/**backward** *edge between two consecutive layers*
  - *non-tree* **cross-edge** *with both* $\mathbf{u}, \mathbf{v}$ *in same layer*
  - $\Longrightarrow$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

BFS with Layers: Properties

For directed graphs

Proposition

The following properties hold on termination of **BFSLayers**$(\mathbf{s})$*, if* $\mathbf{G}$ *is directed.*

For each edge $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ *is one of four types:*

- *a* **tree** *edge between consecutive layers,* $\mathbf{u} \in \mathbf{L_i}, \mathbf{v} \in \mathbf{L_{i+1}}$ *for some* $\mathbf{i} \geq \mathbf{0}$
- *a non-tree* **forward** *edge between consecutive layers*
- *a non-tree* **backward** *edge*
- *a* **cross-edge** *with both* $\mathbf{u}, \mathbf{v}$ *in same layer*

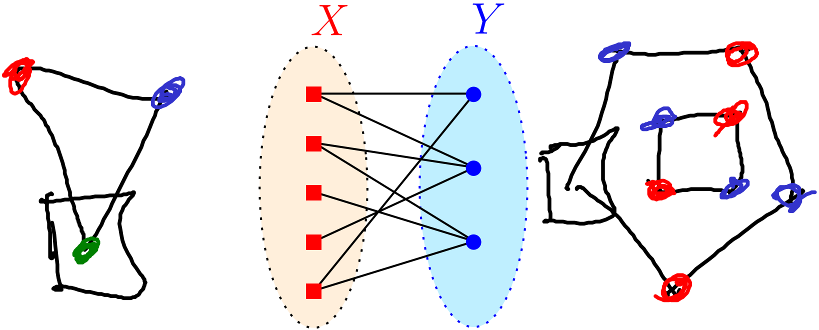
Part II

Bipartite Graphs and an application of BFS

Bipartite Graphs

Definition (Bipartite Graph)

Undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a **bipartite graph** if $\mathbf{V}$ can be partitioned into $\mathbf{X}$ and $\mathbf{Y}$ s.t. all edges in $\mathbf{E}$ are between $\mathbf{X}$ and $\mathbf{Y}$.



Bipartite Graph Characterization

Question

When is a graph bipartite?

Proposition

Every tree is a bipartite graph.

Proof.

Root tree T at some node r. Let L_i be all nodes at level i, that is, L_i is all nodes at distance i from root r. Now define X to be all nodes at even levels and Y to be all nodes at odd level. Only edges in T are between levels. □

Proposition

An odd length cycle is not bipartite.

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Odd Cycles are not Bipartite

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An odd length cycle is not bipartite.

Proof.

Let $\mathbf{C} = \mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_{2k+1}}, \mathbf{u_1}$ be an odd cycle. Suppose $\mathbf{C}$ is a bipartite graph and let $\mathbf{X}, \mathbf{Y}$ be the bipartition. Without loss of generality $\mathbf{u_1} \in \mathbf{X}$. Implies $\mathbf{u_2} \in \mathbf{Y}$. Implies $\mathbf{u_3} \in \mathbf{X}$. Inductively, $\mathbf{u_i} \in \mathbf{X}$ if $\mathbf{i}$ is odd $\mathbf{u_i} \in \mathbf{Y}$ if $\mathbf{i}$ is even. But $\{\mathbf{u_1}, \mathbf{u_{2k+1}}\}$ is an edge and both belong to $\mathbf{X}$! □

Subgraphs

Definition

Given a graph $G = (V, E)$ a **subgraph** of G is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

Proposition

If G is bipartite then any subgraph H of G is also bipartite.

Proposition

A graph G is not bipartite if G has an odd cycle C as a subgraph.

Proof.

If G is bipartite then since C is a subgraph, C is also bipartite (by above proposition). However, C is not bipartite! □

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Bipartite Graph Characterization

Theorem

A graph $\mathbf{G}$ is bipartite if and only if it has no odd length cycle as subgraph.

Proof.

Only If: $\mathbf{G}$ has an odd cycle implies $\mathbf{G}$ is not bipartite.

If: $\mathbf{G}$ has no odd length cycle. Assume without loss of generality that $\mathbf{G}$ is connected.

- Pick $\mathbf{u}$ arbitrarily and do $\mathbf{BFS}(\mathbf{u})$
- $\mathbf{X} = \cup_{\mathbf{i} \text{ is even}} \mathbf{L_i}$ and $\mathbf{Y} = \cup_{\mathbf{i} \text{ is odd}} \mathbf{L_i}$
- **Claim:** $\mathbf{X}$ and $\mathbf{Y}$ is a valid bipartition if $\mathbf{G}$ has no odd length cycle.

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□

Proof of Claim

Claim

In **BFS**$(\mathbf{u})$ *if* $\mathbf{a}, \mathbf{b} \in \mathbf{L_i}$ *and* $(\mathbf{a}, \mathbf{b})$ *is an edge then there is an odd length cycle containing* $(\mathbf{a}, \mathbf{b})$.

Proof.

Let $\mathbf{v}$ be least common ancestor of $\mathbf{a}, \mathbf{b}$ in **BFS** tree $\mathbf{T}$.
$\mathbf{v}$ is in some level $\mathbf{j} < \mathbf{i}$ (could be $\mathbf{u}$ itself).
Path from $\mathbf{v} \rightsquigarrow \mathbf{a}$ in $\mathbf{T}$ is of length $\mathbf{j} - \mathbf{i}$.
Path from $\mathbf{v} \rightsquigarrow \mathbf{b}$ in $\mathbf{T}$ is of length $\mathbf{j} - \mathbf{i}$.
These two paths plus $(\mathbf{a}, \mathbf{b})$ forms an odd cycle of length $2(\mathbf{j} - \mathbf{i}) + 1$. □

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Another tidbit

Corollary

There is an $\mathbf{O(n + m)}$ *time algorithm to check if* $\mathbf{G}$ *is bipartite and output an odd cycle if it is not.*

Part III

Shortest Paths and Dijkstra's Algorithm

Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.
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Many applications!

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Single-Source Shortest Paths:

Non-Negative Edge Lengths

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Input A (undirected or directed) graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with non-negative edge lengths. For edge $\mathbf{e} = (\mathbf{u}, \mathbf{v})$, $\ell(\mathbf{e}) = \ell(\mathbf{u}, \mathbf{v})$ is its length.

- Given nodes $\mathbf{s}, \mathbf{t}$ find shortest path from $\mathbf{s}$ to $\mathbf{t}$.
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- Restrict attention to directed graphs
- Undirected graph problem can be reduced to directed graph problem - how?
  - Given undirected graph $\mathbf{G}$, create a new directed graph $\mathbf{G}'$ by replacing each edge $\{\mathbf{u}, \mathbf{v}\}$ in $\mathbf{G}$ by $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{v}, \mathbf{u})$ in $\mathbf{G}'$.
  - set $\ell(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}, \mathbf{u}) = \ell(\{\mathbf{u}, \mathbf{v}\})$
  - Exercise: show reduction works

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Single-Source Shortest Paths via BFS

Special case: All edge lengths are $\mathbf{1}$.

- Run **BFS**($\mathbf{s}$) to get shortest path distances from s to all other nodes.
- $\mathbf{O(m+n)}$ time algorithm.

Special case: Suppose $\ell(\mathbf{e})$ is an integer for all $\mathbf{e}$?
Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(\mathbf{e}) - \mathbf{1}$ dummy nodes on $\mathbf{e}$

Let $\mathbf{L} = \mathbf{max}_{\mathbf{e}}\, \ell(\mathbf{e})$. New graph has $\mathbf{O(mL)}$ edges and $\mathbf{O(mL+n)}$ nodes. **BFS** takes $\mathbf{O(mL+n)}$ time. Not efficient if $\mathbf{L}$ is large.

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Towards an algorithm

Why does **BFS** work?

BFS(s) explores nodes in increasing distance from s

Lemma

Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

- $s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ *is a shortest path from s to v_i*
- $\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$.

Proof.

Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \to v_1 \to \ldots \to v_i$. Then P' concatenated with $v_i \to v_{i+1} \ldots \to v_k$ contains a strictly shorter

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Let $\mathbf{G}$ *be a directed graph with non-negative edge lengths. Let* $\mathrm{dist}(\mathbf{s}, \mathbf{v})$ *denote the shortest path length from* $\mathbf{s}$ *to* $\mathbf{v}$*. If* $\mathbf{s} = \mathbf{v}_0 \to \mathbf{v}_1 \to \mathbf{v}_2 \to \ldots \to \mathbf{v}_k$ *is a shortest path from* $\mathbf{s}$ *to* $\mathbf{v}_k$ *then for* $1 \leq i < k$*:*

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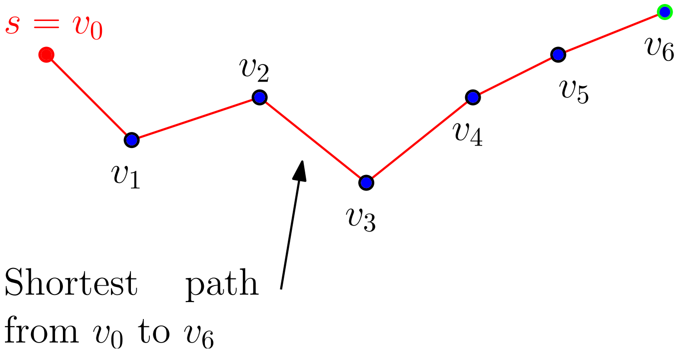
A proof by picture



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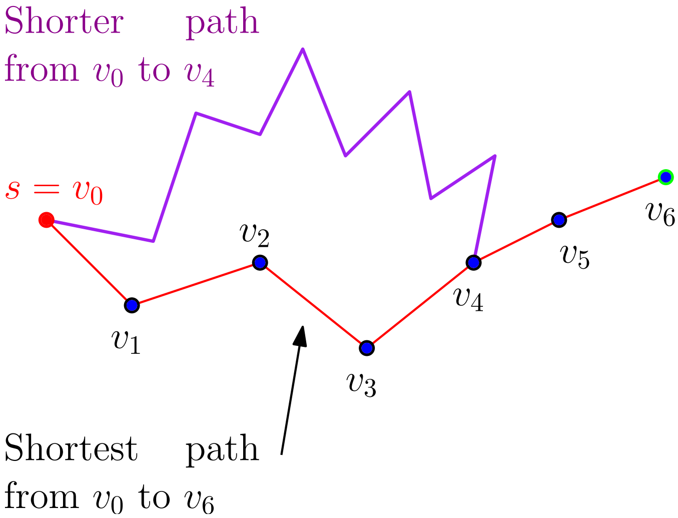

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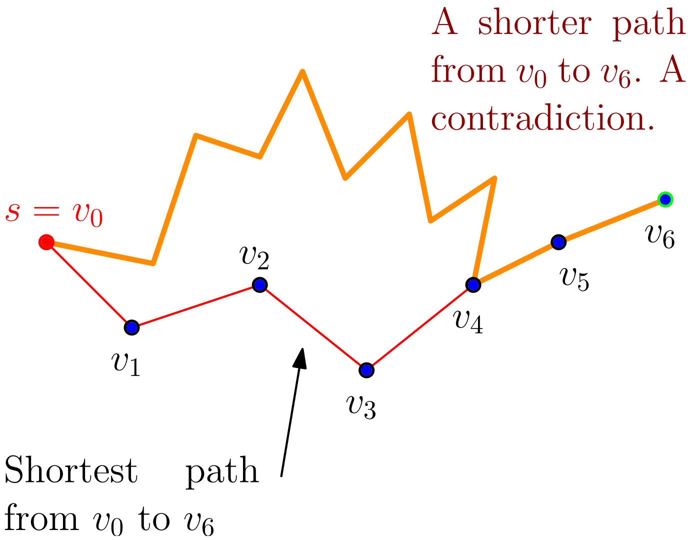

A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

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Initialize for each node v, dist(s,v) = ∞
Initialize S = ∅,
for i = 1 to |V| do
    (* Invariant:  S contains the i − 1 closest nodes to s *)
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How can we implement the step in the for loop?

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Finding the ith closest node

- $\mathbf{S}$ contains the $\mathbf{i} - \mathbf{1}$ closest nodes to $\mathbf{s}$
- Want to find the $\mathbf{i}$th closest node from $\mathbf{V} - \mathbf{S}$.

What do we know about the $\mathbf{i}$th closest node?

Claim

Let $\mathbf{P}$ be a shortest path from $\mathbf{s}$ to $\mathbf{v}$ where $\mathbf{v}$ is the $\mathbf{i}$th closest node. Then, all intermediate nodes in $\mathbf{P}$ belong to $\mathbf{S}$.

Proof.

If $\mathbf{P}$ had an intermediate node $\mathbf{u}$ not in $\mathbf{S}$ then $\mathbf{u}$ will be closer to $\mathbf{s}$ than $\mathbf{v}$. Implies $\mathbf{v}$ is not the $\mathbf{i}$th closest node to $\mathbf{s}$ - recall that $\mathbf{S}$ already has the $\mathbf{i} - \mathbf{1}$ closest nodes. □

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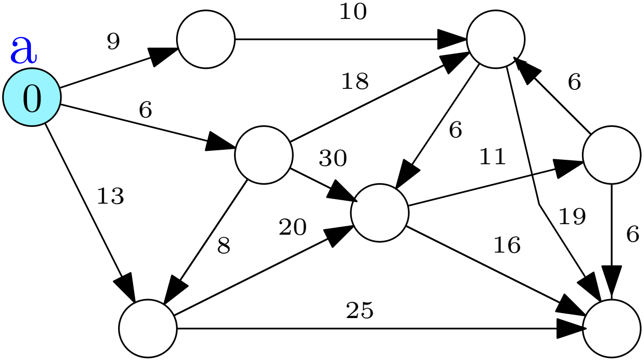
Finding the ith closest node repeatedly

An example



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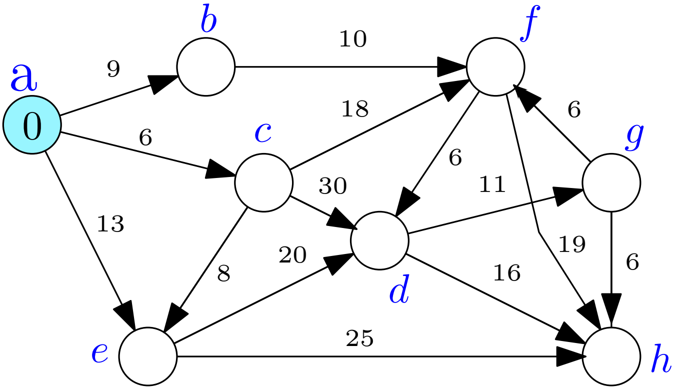

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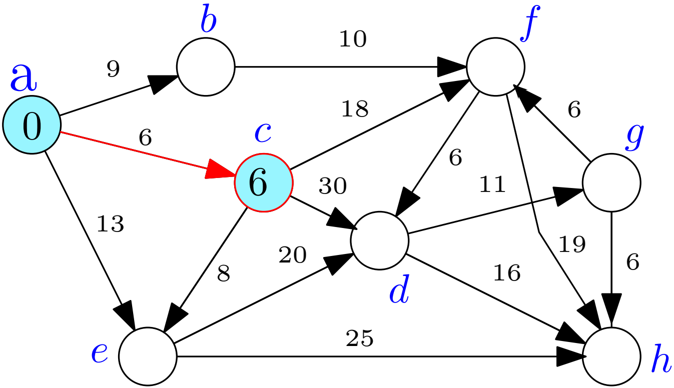

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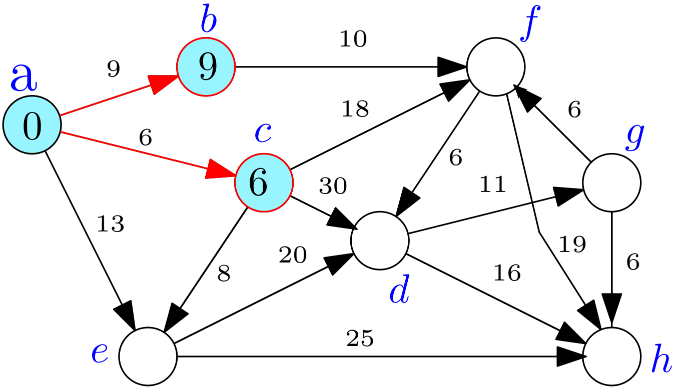

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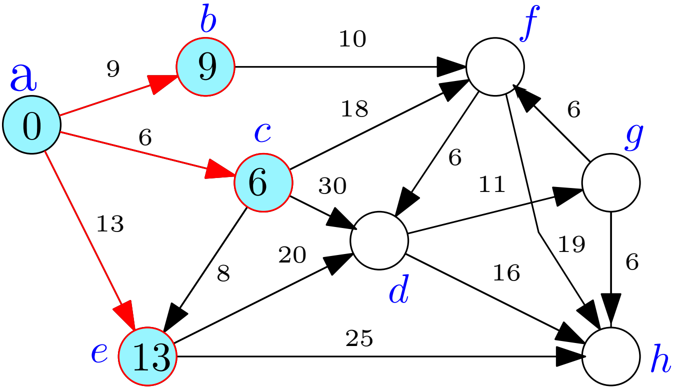

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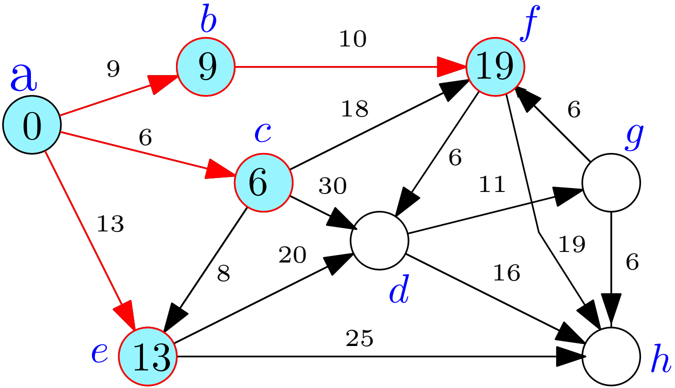

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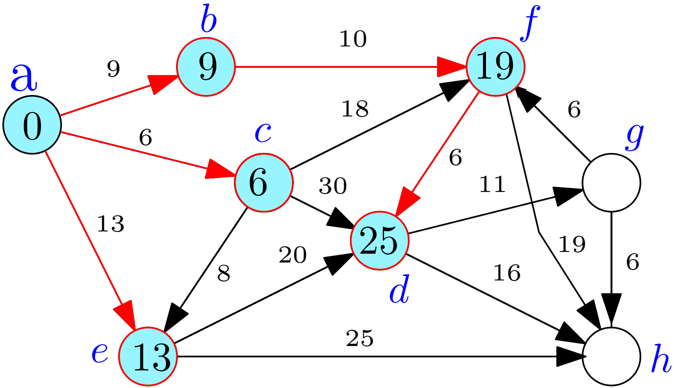

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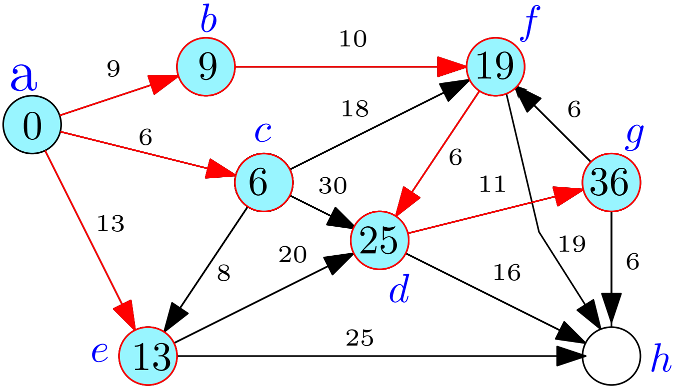

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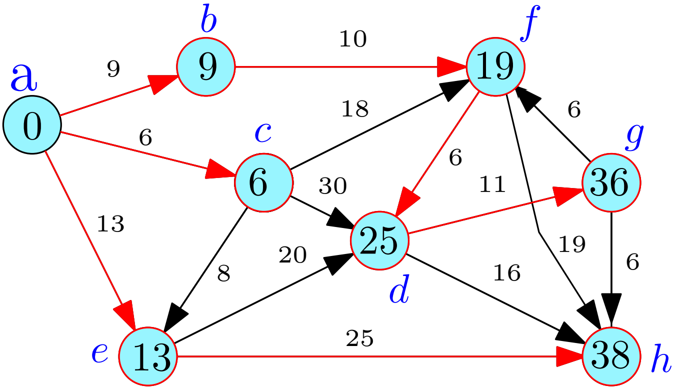

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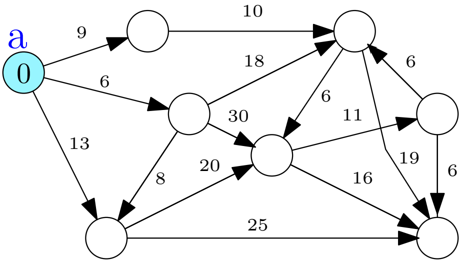

Corollary

The ith closest node is adjacent to S.

Finding the ith closest node

- $\mathbf{S}$ contains the $\mathbf{i-1}$ closest nodes to $\mathbf{s}$
- Want to find the $\mathbf{i}$th closest node from $\mathbf{V-S}$.

- For each $\mathbf{u \in V - S}$ let $\mathbf{P(s, u, S)}$ be a shortest path from $\mathbf{s}$ to $\mathbf{u}$ using only nodes in $\mathbf{S}$ as intermediate vertices.
- Let $\mathbf{d'(s, u)}$ be the length of $\mathbf{P(s, u, S)}$

Observations: for each $\mathbf{u \in V - S}$,

- $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \leq \mathbf{d'(s, u)}$ since we are constraining the paths
- $\mathbf{d'(s, u)} = \mathbf{min}_{\mathbf{a \in S}}(\mathrm{dist}(\mathbf{s}, \mathbf{a}) + \ell(\mathbf{a}, \mathbf{u}))$ - Why?

Lemma

If $\mathbf{v}$ *is the* $\mathbf{i}$*th closest node to* $\mathbf{s}$*, then* $\mathbf{d'(s, v)} = \mathrm{dist}(\mathbf{s}, \mathbf{v})$.

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Proof.

Let $\mathbf{v}$ be the $\mathbf{i}$th closest node to $\mathbf{s}$. Then there is a shortest path $\mathbf{P}$ from $\mathbf{s}$ to $\mathbf{v}$ that contains only nodes in $\mathbf{S}$ as intermediate nodes (see previous claim). Therefore $\mathbf{d'(s,v)} = \mathrm{dist}(\mathbf{s},\mathbf{v})$. □

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Corollary

The $\mathbf{i}$*th closest node to* $\mathbf{s}$ *is the node* $\mathbf{v} \in \mathbf{V} - \mathbf{S}$ *such that* $\mathbf{d'(s, v)} = \min_{\mathbf{u} \in \mathbf{V} - \mathbf{S}} \mathbf{d'(s, u)}$.

Proof.

For every node $\mathbf{u} \in \mathbf{V} - \mathbf{S}$, $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \leq \mathbf{d'(s, u)}$ and for the $\mathbf{i}$th closest node $\mathbf{v}$, $\mathrm{dist}(\mathbf{s}, \mathbf{v}) = \mathbf{d'(s, v)}$. Moreover, $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \geq \mathrm{dist}(\mathbf{s}, \mathbf{v})$ for each $\mathbf{u} \in \mathbf{V} - \mathbf{S}$. □

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    for each node u in V \ S
        compute d'(s,u) = min_{a∈S} (dist(s,a) + ℓ(a,u))
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Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

- n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in S; $O(m + n)$ time/iteration.

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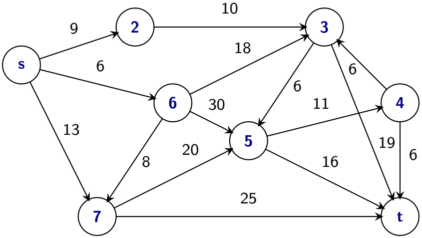
    Let v be such that d'(s,v) = min_{u∈V−S} d'(s,u)
    dist(s,v) = d'(s,v)
    S = S ∪ {v}
    for each node u in V \ S
        compute d'(s,u) = min_{a∈S} (dist(s,a) + ℓ(a,u))
```

Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

- n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in S; $O(m + n)$ time/iteration.

Example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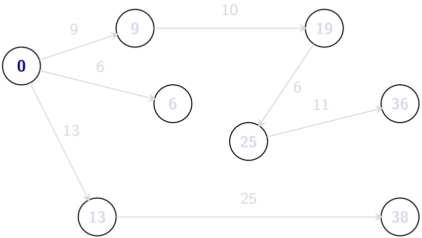
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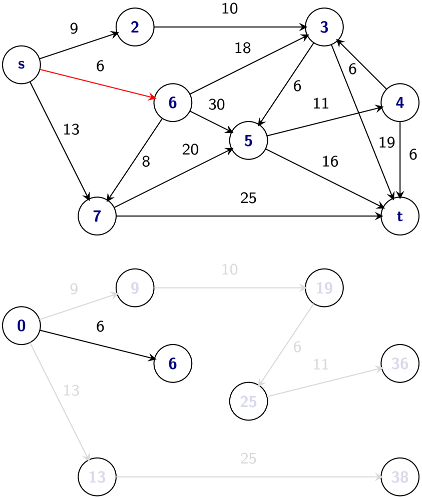
Example



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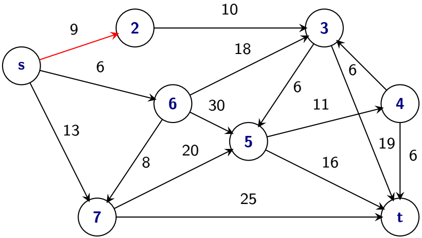

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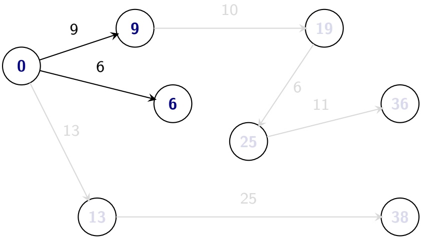

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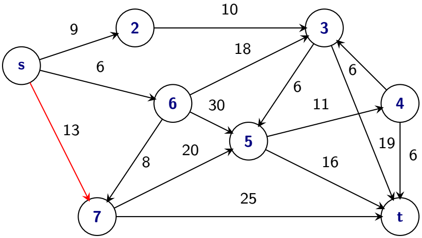

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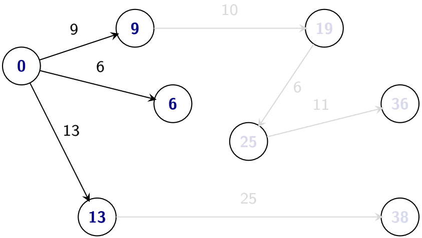

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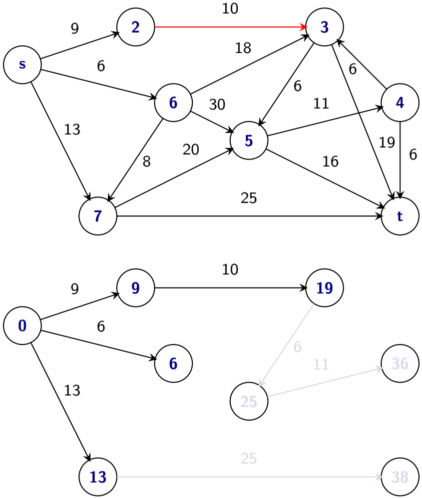
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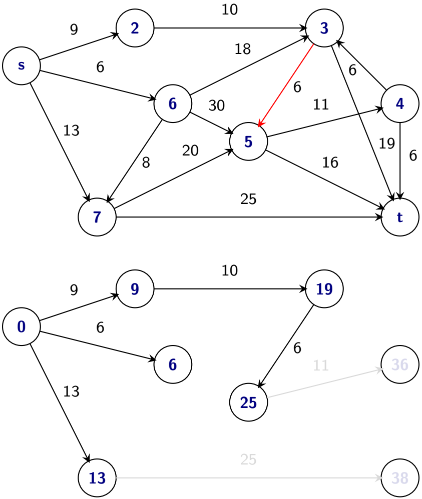

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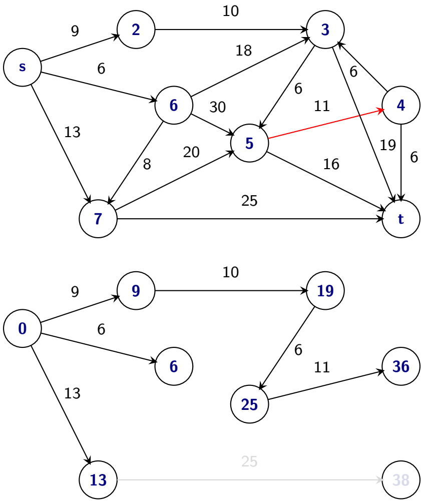

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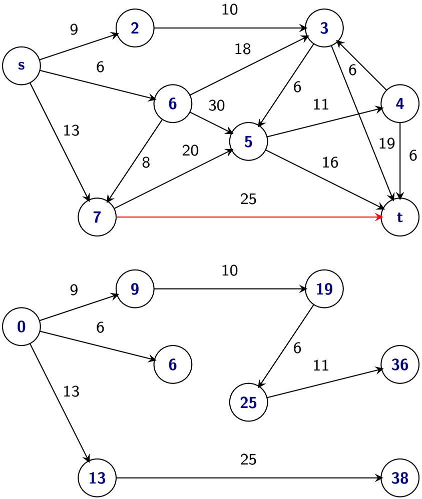

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Improved Algorithm

- Main work is to compute the $\mathbf{d'(s,u)}$ values in each iteration
- $\mathbf{d'(s,u)}$ changes from iteration $\mathbf{i}$ to $\mathbf{i+1}$ only because of the node $\mathbf{v}$ that is added to $\mathbf{S}$ in iteration $\mathbf{i}$.

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Initialize for each node v, dist(s,v) = d'(s,v) = ∞
Initialize S = ∅, d'(s,s) = 0
for i = 1 to |V| do
    // S contains the i − 1 closest nodes to s,
    //        and the values of d'(s,u) are current
    Let v be such that d'(s,v) = min_{u∈V−S} d'(s,u)
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    S = S ∪ {v}
    Update d'(s,u) for each u in V-S as follows:
        d'(s,u) = min(d'(s,u), dist(s,v) + ℓ(v,u))
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Running time: $\mathbf{O(m+n^2)}$ time.

- $\mathbf{n}$ outer iterations and in each iteration following steps
- updating $\mathbf{d'(s,u)}$ after $\mathbf{v}$ added takes $\mathbf{O(deg(v))}$ time so total

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Running time: $O(m + n^2)$ time.

- n outer iterations and in each iteration following steps
- updating $d'(s, u)$ after v added takes $O(deg(v))$ time so total work is $O(m)$ since a node enters S only once
- Finding v from $d'(s, u)$ values is $O(n)$ time

Dijkstra's Algorithm

- eliminate $\mathbf{d}'(\mathbf{s},\mathbf{u})$ and let $\mathrm{dist}(\mathbf{s},\mathbf{u})$ maintain it
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Priority Queues to maintain **dist** values for faster running time

- Using heaps and standard priority queues: $O((m+n)\log n)$
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Priority Queues

Data structure to store a set $\mathbf{S}$ of $\mathbf{n}$ elements where each element $\mathbf{v} \in \mathbf{S}$ has an associated real/integer key $\mathbf{k(v)}$ such that the following operations

- `makeQ`: create an empty queue
- `findMin`: find the minimum key in $\mathbf{S}$
- `extractMin`: Remove $\mathbf{v} \in \mathbf{S}$ with smallest key and return it
- `add(v, k(v))`: Add new element $\mathbf{v}$ with key $\mathbf{k(v)}$ to $\mathbf{S}$
- `delete(v)`: Remove element $\mathbf{v}$ from $\mathbf{S}$
- `decreaseKey(v, k'(v))`: *decrease* key of $\mathbf{v}$ from $\mathbf{k(v)}$ (current key) to $\mathbf{k'(v)}$ (new key). Assumption: $\mathbf{k'(v)} \leq \mathbf{k(v)}$
- `meld`: merge two separate priority queues into one

can be performed in $\mathbf{O(\log n)}$ time each.

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Dijkstra's Algorithm using Priority Queues

```
Q = makePQ()
insert(Q, (s,0))
for each node u ≠ s do
    insert(Q, (u,∞))
S = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    S = S ∪ {v}
    For each u in Adj(v) do
        decreaseKey(Q, (u, min(dist(s,u), dist(s,v) + ℓ(v,u))))
```

Priority Queue operations:

- $O(n)$ insert operations
- $O(n)$ extractMin operations
- $O(m)$ decreaseKey operations

Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

- All operations can be done in $O(\log n)$ time

Dijkstra's algorithm can be implemented in $O((n + m) \log n)$ time.

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Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

- `extractMin`, `add`, `delete`, `meld` in $\mathbf{O(\log n)}$ time
- `decreaseKey` in $\mathbf{O(1)}$ *amortized* time: ℓ decreaseKey operations for $\ell \geq \mathbf{n}$ take *together* $\mathbf{O(\ell)}$ time
- Relaxed Heaps: `decreaseKey` in $\mathbf{O(1)}$ worst case time but at the expense of `meld` (not necessary for Dijkstra's algorithm)

— Dijkstra's algorithm can be implemented in $\mathbf{O(n \log n + m)}$ time. If $\mathbf{m = \Omega(n \log n)}$, running time is linear in input size.

— Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

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Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to **V**.

Question: How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s, 0))
prev(s) = null
for each node u ≠ s do
    insert(Q, (u, ∞) )
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S = ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    S = S ∪ {v}
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```

Shortest Path Tree

Lemma

The edge set $(\mathbf{u}, \mathrm{prev}(\mathbf{u}))$ *is the* reverse *of a shortest path tree rooted at* $\mathbf{s}$. *For each* $\mathbf{u}$, *the reverse of the path from* $\mathbf{u}$ *to* $\mathbf{s}$ *in the tree is a shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$.

Proof Sketch.

- The edgeset $\{(\mathbf{u}, \mathrm{prev}(\mathbf{u})) \mid \mathbf{u} \in \mathbf{V}\}$ induces a directed in-tree rooted at $\mathbf{s}$ (Why?)
- Use induction on $|\mathbf{S}|$ to argue that the tree is a shortest path tree for nodes in $\mathbf{V}$.

□

Shortest paths to $\mathbf{s}$

Dijkstra's algorithm gives shortest paths from $\mathbf{s}$ to all nodes in $\mathbf{V}$.

How do we find shortest paths from all of $\mathbf{V}$ to $\mathbf{s}$?

- In undirected graphs shortest path from $\mathbf{s}$ to $\mathbf{u}$ is a shortest path from $\mathbf{u}$ to $\mathbf{s}$ so there is no need to distinguish.
- In directed graphs, use Dijkstra's algorithm in $\mathbf{G}^{\mathbf{rev}}$!

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