NP Completeness and Cook-Levin Theorem

Lecture 22
April 19, 2011
P and NP and Turing Machines

- **P**: set of decision problems that have polynomial time algorithms.
- **NP**: set of decision problems that have polynomial time non-deterministic algorithms.

**Question:** What is an algorithm? Depends on the model of computation!

What is our model of computation?

Formally speaking our model of computation is Turing Machines.
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Turing Machines: Recap

- Infinite tape.
- Finite state control.
- Input at beginning of tape.
- Special tape letter “blank” ☐.
- Head can move only one cell to left or right.
Turing Machines: Formally

A Turing Machine $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$:

- $Q$ is set of states in finite control
- $q_0$ start state, $q_{\text{accept}}$ is accept state, $q_{\text{reject}}$ is reject state
- $\Sigma$ is input alphabet, $\Gamma$ is tape alphabet (includes $\square$)
- $\delta : Q \times \Gamma \rightarrow \{L, R\} \times \Gamma \times Q$ is transition function
  - $\delta(q, a) = (q', b, L)$ means that $M$ in state $q$ and head seeing $a$ on tape will move to state $q'$ while replacing $a$ on tape with $b$ and head moves left.

$L(M)$: language accepted by $M$ is set of all input strings $s$ on which $M$ accepts; that is:

- $TM$ is started in state $q_0$.
- Initially, the tape head is located at the first cell.
- The tape contain $s$ on the tape followed by blanks.
- The $TM$ halts in the state $q_{\text{accept}}$.
Definition

$M$ is a polynomial time $\text{TM}$ if there is some polynomial $p(\cdot)$ such that on all inputs $w$, $M$ halts in $p(|w|)$ steps.

Definition

$L$ is a language in $\text{P}$ iff there is a polynomial time $\text{TM}$ $M$ such that $L = L(M)$. 
Definition

$L$ is an $\mathbf{NP}$ language iff there is a non-deterministic polynomial time TM $M$ such that $L = L(M)$.

Non-deterministic TM: each step has a choice of moves

- $\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$.

- Example: $\delta(q, a) = \{(q_1, b, L), (q_2, c, R), (q_3, a, R)\}$ means that $M$ can non-deterministically choose one of the three possible moves from $(q, a)$.

- $L(M)$: set of all strings $s$ on which there exists some sequence of valid choices at each step that lead from $q_0$ to $q_{\text{accept}}$. 
**NP via TMs**

**Definition**

$L$ is an **NP** language iff there is a *non-deterministic* polynomial time **TM** $M$ such that $L = L(M)$.

Non-deterministic **TM**: each step has a choice of moves

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2. Example: $\delta(q, a) = \{(q_1, b, L), (q_2, c, R), (q_3, a, R)\}$ means that $M$ can non-deterministically choose one of the three possible moves from $(q, a)$.

$L(M)$: set of all strings $s$ on which there exists some sequence of valid choices at each step that lead from $q_0$ to $q_{accept}$
Non-deterministic TMs vs certifiers

NP

Two definition of **NP**:

- **L** is in **NP** iff **L** has a polynomial time certifier **C(·, ·)**.
- **L** is in **NP** iff **L** is decided by a non-deterministic polynomial time **TM M**.

**Claim**

Two definitions are equivalent.

Why?

Informal proof idea: the certificate **t** for **C** corresponds to non-deterministic choices of **M** and vice-versa.

In other words **L** is in **NP** iff **L** is accepted by a **NTM** which first guesses a proof **t** of length poly in input |**s**| and then acts as a deterministic **TM**.
Non-determinism, guessing and verification

- A non-deterministic machine has choices at each step and accepts a string if there exists a set of choices which lead to a final state.
- Equivalently the choices can be thought of as guessing a solution and then verifying that solution. In this view all the choices are made a priori and hence the verification can be deterministic. The “guess” is the “proof” and the “verifier” is the “certifier”.
- We reemphasize the asymmetry inherent in the definition of non-determinism. Strings in the language can be easily verified. No easy way to verify that a string is not in the language.
Why do we use TMs some times and RAM Model other times?

- **TMs** are very simple: no complicated instruction set, no jumps/pointers, no explicit loops etc.
  - Simplicity is useful in proofs.
  - The “right” formal bare-bones model when dealing with subtleties.

- **RAM** model is a closer approximation to the running time/space usage of realistic computers for reasonable problem sizes
  - Not appropriate for certain kinds of formal proofs when algorithms can take super-polynomial time and space
“Hardest” Problems

**Question**
What is the hardest problem in $\text{NP}$? How do we define it?

**Towards a definition**
- Hardest problem must be in $\text{NP}$.
- Hardest problem must be at least as “difficult” as every other problem in $\text{NP}$. 
NP-Complete Problems

**Definition**

A problem $X$ is said to be **NP-Complete** if

- $X \in \text{NP}$, and
- (Hardness) For any $Y \in \text{NP}$, $Y \leq_P X$. 
Proposition

Suppose $X$ is NP-Complete. Then $X$ can be solved in polynomial time if and only if $P = NP$.

Proof.

$\Rightarrow$ Suppose $X$ can be solved in polynomial time

- Let $Y \in NP$. We know $Y \leq_p X$.
- We showed that if $Y \leq_p X$ and $X$ can be solved in polynomial time, then $Y$ can be solved in polynomial time.
- Thus, every problem $Y \in NP$ is such that $Y \in P$; $NP \subseteq P$.
- Since $P \subseteq NP$, we have $P = NP$.

$\Leftarrow$ Since $P = NP$, and $X \in NP$, we have a polynomial time algorithm for $X$. 

$\square$
A problem $X$ is said to be NP-Hard if

- (Hardness) For any $Y \in \text{NP}$, $Y \leq_P X$

An NP-Hard problem need not be in NP!

Example: Halting problem is NP-Hard (why?) but not NP-Complete.
Consequences of proving \textbf{NP-Completeness}

If $X$ is \textbf{NP-Complete}
  
  - Since we believe $P \neq NP$,
  - and solving $X$ implies $P = NP$.

$X$ is \textbf{unlikely} to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for $X$.

(This is proof by mob opinion --- take with a grain of salt.)
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**NP-Complete Problems**

**Question**
Are there any problems that are NP-Complete?

**Answer**
Yes! Many, many problems are NP-Complete.
A circuit is a directed *acyclic* graph with

- **Input** vertices (without incoming edges) labelled with 0, 1 or a distinct variable
- Every other vertex is labelled \( \lor, \land \) or \( \neg \)
- Single node **output** vertex with no outgoing edges
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A circuit is a directed acyclic graph with

- **Input** vertices (without incoming edges) labelled with 0, 1 or a distinct variable
- Every other vertex is labelled ∨, ∧ or ¬
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Cook-Levin Theorem

Definition (Circuit Satisfaction (CSAT).)

Given a circuit as input, is there an assignment to the input variables that causes the output to get value 1?

Theorem (Cook-Levin)

CSAT is NP-Complete.

Need to show

- CSAT is in NP
- every NP problem X reduces to CSAT.
Claim

\textbf{CSAT} is in \textbf{NP}.

- **Certificate:** Assignment to input variables.
- **Certifier:** Evaluate the value of each gate in a topological sort of \textbf{DAG} and check the output gate value.
Claim

**CSAT** is in **NP**.

- **Certificate**: Assignment to input variables.
- **Certifier**: Evaluate the value of each gate in a topological sort of **DAG** and check the output gate value.
CSAT is \textbf{NP}-hard: Idea

Need to show that every \textbf{NP} problem $X$ reduces to \textbf{CSAT}.

What does it mean that $X \in \textbf{NP}$?

$X \in \textbf{NP}$ implies that there are polynomials $p()$ and $q()$ and certifier/verifier program $C$ such that for every string $s$ the following is true:

- If $s$ is a YES instance ($s \in X$) then there is a \textit{proof} $t$ of length $p(|s|)$ such that $C(s, t)$ says YES.
- If $s$ is a NO instance ($s \not\in X$) then for every string $t$ of length at $p(|s|)$, $C(s, t)$ says NO.
- $C(s, t)$ runs in time $q(|s| + |t|)$ time (hence polynomial time).
Reducing $X$ to $\text{CSAT}$

$X$ is in $\text{NP}$ means we have access to $p(), q(), C(\cdot, \cdot)$.

What is $C(\cdot, \cdot)$? It is a program or equivalently a Turing Machine!

How are $p()$ and $q()$ given? As numbers.

Example: if 3 is given then $p(n) = n^3$.

Thus an $\text{NP}$ problem is essentially a three tuple $< p, q, C >$ where $C$ is either a program or a $\text{TM}$.
Reducing $X$ to $\text{CSAT}$

Thus an $\text{NP}$ problem is essentially a three tuple $< p, q, C >$ where $C$ is either a program or $\text{TM}$.

**Problem X:** Given string $s$, is $s \in X$?

Same as the following: is there a proof $t$ of length $p(|s|)$ such that $C(s, t)$ says YES.

How do we reduce $X$ to $\text{CSAT}$? Need an algorithm $\mathcal{A}$ that

- takes $s$ (and $< p, q, C >$) and creates a circuit $G$ in polynomial time in $|s|$ (note that $< p, q, C >$ are fixed).
- $G$ is satisfiable if and only if there is a proof $t$ such that $C(s, t)$ says YES.
Reducing \textbf{X} to \textbf{CSAT}

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Simple but Big Idea: Programs are essentially the same as Circuits!

- Convert \( C(s, t) \) into a circuit \( G \) with \( t \) as unknown inputs (rest is known including \( s \))
- We know that \(|t| = p(|s|)\) so express boolean string \( t \) as \( p(|s|) \) variables \( t_1, t_2, \ldots, t_k \) where \( k = p(|s|) \).
- Asking if there is a proof \( t \) that makes \( C(s, t) \) say YES is same as whether there is an assignment of values to “unknown” variables \( t_1, t_2, \ldots, t_k \) that will make \( G \) evaluate to true/YES.
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Example: **Independent Set**

- **Problem:** Does $G = (V, E)$ have an Independent Set of size $\geq k$?
  - **Certificate:** Set $S \subseteq V$
  - **Certifier:** Check $|S| \geq k$ and no pair of vertices in $S$ is connected by an edge

Formally, why is **Independent Set** in **NP**?
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Example: **Independent Set**

Formally why is **Independent Set** in **NP**?

- **Input:**
  \[ < n, y_{1,1}, y_{1,2}, \ldots, y_{1,n}, y_{2,1}, \ldots, y_{2,n}, \ldots, y_{n,1}, \ldots, y_{n,n}, k > \]
  encodes \[ < G, k > \].
  - \( n \) is number of vertices in \( G \)
  - \( y_{i,j} \) is a bit which is 1 if edge \((i, j)\) is in \( G \) and 0 otherwise (adjacency matrix representation)
  - \( k \) is size of independent set.

- **Certificate:** \( t = t_1 t_2 \ldots t_n \). Interpretation is that \( t_i \) is 1 if vertex \( i \) is in the independent set, 0 otherwise.
Certifier for Independent Set

Certifier $C(s, t)$ for Independent Set:

if $(t_1 + t_2 + \ldots + t_n < k)$ then
    return NO
else
    for each $(i, j)$ do
        if $(t_i \land t_j \land y_{i,j})$ then
            return NO
    return YES
Example: Independent Set

Figure: Graph $G$ with $k = 2$
Consider “program” $A$ that takes $f(|s|)$ steps on input string $s$.

**Question:** What computer is the program running on and what does *step* mean?

Real computers difficult to reason with mathematically because

- instruction set is too rich
- pointers and control flow jumps in one step
- assumption that pointer to code fits in one word

**Turing Machines**

- simpler model of computation to reason with
- can simulate real computers with *polynomial* slow down
- all moves are *local* (head moves only one cell)
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**Turing Machines**

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Assume $C(\cdot, \cdot)$ is a (deterministic) Turing Machine $M$

**Problem:** Given $M$, input $s$, $p$, $q$ decide if there is a proof $t$ of length $p(|s|)$ such that $M$ on $s$, $t$ will halt in $q(|s|)$ time and say YES.

There is an algorithm $A$ that can reduce above problem to **CSAT** mechanically as follows.

- $A$ first computes $p(|s|)$ and $q(|s|)$.
- Knows that $M$ can use at most $q(|s|)$ memory/tape cells
- Knows that $M$ can run for at most $q(|s|)$ time
- Simulates the evolution of the state of $M$ and memory over time using a big circuit.
Think of $M$’s state at time $\ell$ as a string $x^\ell = x_1 x_2 \ldots x_k$ where each $x_i \in \{0, 1, B\} \times Q \cup \{q_{-1}\}$.

At time 0 the state of $M$ consists of input string $s$ a guess $t$ (unknown variables) of length $p(|s|)$ and rest $q(|s|)$ blank symbols.

At time $q(|s|)$ we wish to know if $M$ stops in $q_{\text{accept}}$ with say all blanks on the tape.

We write a circuit $C_\ell$ which captures the transition of $M$ from time $\ell$ to time $\ell + 1$.

Composition of the circuits for all times 0 to $q(|s|)$ gives a big (still poly) sized circuit $C$.

The final output of $C$ should be true if and only if the entire state of $M$ at the end leads to an accept state.
NP-Hardness of Circuit Satisfaction

Key Ideas in reduction:

- Use **TM**s as the code for certifier for simplicity
- Since \( p() \) and \( q() \) are known to \( A \), it can set up all required memory and time steps in advance
- Simulate computation of the **TM** from one time to the next as a circuit that only looks at three adjacent cells at a time

Note: Above reduction can be done to **SAT** as well. Reduction to **SAT** was the original proof of Steve Cook.
NP-Hardness of Circuit Satisfaction

Key Ideas in reduction:

- Use TMs as the code for certifier for simplicity
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Note: Above reduction can be done to SAT as well. Reduction to SAT was the original proof of Steve Cook.
SAT is NP-Complete

- We have seen that \( \text{SAT} \in \text{NP} \)
- To show \textbf{NP-Hardness}, we will reduce Circuit Satisfiability (\textsc{CSAT}) to \textsc{SAT}

Instance of \textsc{CSAT} (we label each node):

\[
\begin{align*}
\text{Inputs:} & \quad 1, a, \ ?b, \ ?c, \ 0d, \ ?e \\
\text{Output:} & \quad \wedge, k
\end{align*}
\]
Converting a circuit into a CNF formula

Label the nodes

(A) Input circuit

(B) Label the nodes.
Converting a circuit into a \textbf{CNF} formula

Introduce a variable for each node

(B) Label the nodes.

(C) Introduce var for each node.
Converting a circuit into a **CNF** formula

Write a sub-formula for each variable that is true if the var is computed correctly.

(C) Introduce var for each node.

\( x_k \)  (Demand a sat’ assignment!)
\[ x_k = x_i \land x_k \]
\[ x_j = x_g \land x_h \]
\[ x_i = \neg x_f \]
\[ x_h = x_d \lor x_e \]
\[ x_g = x_b \lor x_c \]
\[ x_f = x_a \land x_b \]
\[ x_d = 0 \]
\[ x_a = 1 \]

(D) Write a sub-formula for each variable that is true if the var is computed correctly.
Converting a circuit into a **CNF** formula

Convert each sub-formula to an equivalent CNF formula

<table>
<thead>
<tr>
<th>$x_k$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$x_k = x_i \land x_j$</td>
<td>$(\neg x_k \lor x_i) \land (\neg x_k \lor x_j) \land (x_k \lor \neg x_i \lor \neg x_j)$</td>
</tr>
<tr>
<td>$x_j = x_g \land x_h$</td>
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</tr>
<tr>
<td>$x_i = \neg x_f$</td>
<td>$(x_i \lor x_f) \land (\neg x_i \lor x_f) \land$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$x_f = x_a \land x_b$</td>
<td>$(\neg x_f \lor x_a) \land (\neg x_f \lor x_b) \land (x_f \lor \neg x_a \lor \neg x_b)$</td>
</tr>
<tr>
<td>$x_d = 0$</td>
<td>$\neg x_d$</td>
</tr>
<tr>
<td>$x_a = 1$</td>
<td>$x_a$</td>
</tr>
</tbody>
</table>
Converting a circuit into a **CNF** formula

**Take the conjunction of all the CNF sub-formulas**

![Circuit Diagram]

\[
x_k \land (\neg x_k \lor x_i) \land (\neg x_k \lor x_j) \\
\land (x_k \lor \neg x_i \lor \neg x_j) \land (\neg x_j \lor x_g) \\
\land (\neg x_j \lor x_h) \land (x_j \lor \neg x_g \lor \neg x_h) \\
\land (x_i \lor x_f) \land (\neg x_i \lor x_f) \\
\land (x_h \lor \neg x_d) \land (x_h \lor \neg x_e) \\
\land (\neg x_h \lor x_d \lor x_e) \land (x_g \lor \neg x_b) \\
\land (x_g \lor \neg x_c) \land (\neg x_g \lor x_b \lor x_c) \\
\land (\neg x_f \lor x_a) \land (\neg x_f \lor x_b) \\
\land (x_f \lor \neg x_a \lor \neg x_b) \land (\neg x_d) \land x_a
\]

We got a **CNF** formula that is satisfiable if and only if the original circuit is satisfiable.
For each gate (vertex) \( v \) in the circuit, create a variable \( x_v \)

Case \( \neg \): \( v \) is labeled \( \neg \) and has one incoming edge from \( u \) (so \( x_v = \neg x_u \)). In SAT formula generate, add clauses \((x_u \lor x_v)\), \((\neg x_u \lor \neg x_v)\). Observe that

\[ x_v = \neg x_u \text{ is true } \iff (x_u \lor x_v) \land (\neg x_u \lor \neg x_v) \text{ both true.} \]
Case $\lor$: So $x_v = x_u \lor x_w$. In SAT formula generated, add clauses $(x_v \lor \neg x_u)$, $(x_v \lor \neg x_w)$, and $(\neg x_v \lor x_u \lor x_w)$. Again, observe that

$$x_v = x_u \lor x_w \text{ is true } \iff (x_v \lor \neg x_u), \quad (x_v \lor \neg x_w), \quad \text{all true.} \quad (\neg x_v \lor x_u \lor x_w)$$
Case $\wedge$: So $x_v = x_u \land x_w$. In SAT formula generated, add clauses $(\neg x_v \lor x_u)$, $(\neg x_v \lor x_w)$, and $(x_v \lor \neg x_u \lor \neg x_w)$. Again observe that

$$x_v = x_u \land x_w \text{ is true} \iff (\neg x_v \lor x_u), (\neg x_v \lor x_w), (x_v \lor \neg x_u \lor \neg x_w) \text{ all true.}$$
If $v$ is an input gate with a fixed value then we do the following. If $x_v = 1$ add clause $x_v$. If $x_v = 0$ add clause $\neg x_v$.

- Add the clause $x_v$ where $v$ is the variable for the output gate.
Correctness of Reduction

Need to show circuit $C$ is satisfiable iff $\varphi_C$ is satisfiable

⇒ Consider a satisfying assignment $a$ for $C$
  - Find values of all gates in $C$ under $a$
  - Give value of gate $v$ to variable $x_v$; call this assignment $a'$
  - $a'$ satisfies $\varphi_C$ (exercise)

⇐ Consider a satisfying assignment $a$ for $\varphi_C$
  - Let $a'$ be the restriction of $a$ to only the input variables
  - Value of gate $v$ under $a'$ is the same as value of $x_v$ in $a$
  - Thus, $a'$ satisfies $C$

Theorem

$SAT$ is $NP$-Complete.
Proving that a problem $X$ is **NP-Complete**

To prove $X$ is **NP-Complete**, show

- Show $X$ is in **NP**.
  - certificate/proof of polynomial size in input
  - polynomial time certifier $C(s, t)$

- Reduction from a known **NP-Complete** problem such as **CSAT** or **SAT** to $X$

SAT $\leq_p X$ implies that every **NP** problem $Y \leq_p X$. Why?

Transitivity of reductions:

$Y \leq_p SAT$ and $SAT \leq_p X$ and hence $Y \leq_p X$. 

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CS473
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NP-Completeness via Reductions

- **CSAT** is **NP-Complete**.
- **CSAT \( \leq_p SAT \)** and **SAT** is in **NP** and hence SAT is **NP-Complete**.
- **SAT \( \leq_p 3\text{-SAT} \)** and hence 3-SAT is **NP-Complete**.
- **3-SAT \( \leq_p \text{Independent Set} \)** (which is in **NP**) and hence **Independent Set** is **NP-Complete**.
- **Vertex Cover** is **NP-Complete**.
- **Clique** is **NP-Complete**.

Hundreds and thousands of different problems from many areas of science and engineering have been shown to be **NP-Complete**.

A surprisingly frequent phenomenon!
NP-Completeness via Reductions

- **CSAT** is NP-Complete.
- **CSAT \(\leq_p SAT\)** and **SAT** is in NP and hence SAT is NP-Complete.
- **SAT \(\leq_p 3\text{-SAT}\)** and hence 3-SAT is NP-Complete.
- **3\text{-SAT} \(\leq_p\) Independent Set (which is in NP) and hence Independent Set is NP-Complete.
- Vertex Cover is NP-Complete.
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Hundreds and thousands of different problems from many areas of science and engineering have been shown to be NP-Complete.

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