NP Completeness and Cook-Levin Theorem

Lecture 22

April 19, 2011

P and NP and Turing Machines

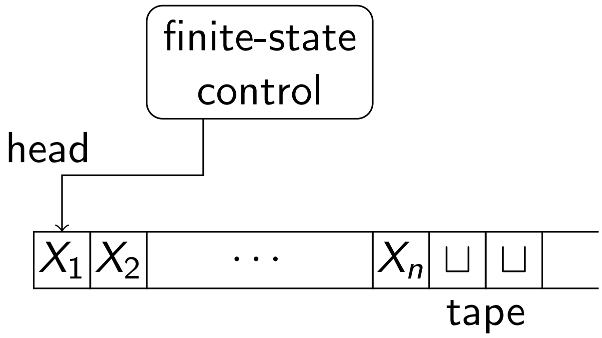
- P: set of decision problems that have polynomial time algorithms.
- NP: set of decision problems that have polynomial time non-deterministic algorithms.

Question: What is an algorithm? Depends on the model of computation!

What is our model of computation?

Formally speaking our model of computation is Turing Machines.

Turing Machines: Recap



- Infinite tape.
- Finite state control.
- Input at beginning of tape.
- Special tape letter "blank" $\sqcup$.
- Head can move only one cell to left or right.

Turing Machines: Formally

A TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$:

- Q is set of states in finite control
- q_0 start state, q_{accept} is accept state, q_{reject} is reject state
- Σ is input alphabet, Γ is tape alphabet (includes $\sqcup$)
- $\delta : Q \times \Gamma \to \{L, R\} \times \Gamma \times Q$ is transition function
  - $\delta(q, a) = (q', b, L)$ means that M in state q and head seeing a on tape will move to state q' while replacing a on tape with b and head moves left.

$L(M)$: language accepted by M is set of all input strings s on which M accepts; that is:

- TM is started in state q_0.
- Initially, the tape head is located at the first cell.
- The tape contain s on the tape followed by blanks.
- The TM halts in the state q_{accept}.

P via TMs

Definition

M is a polynomial time TM if there is some polynomial $p(\cdot)$ such that on all inputs w, M halts in $p(|w|)$ steps.

Definition

L is a language in P iff there is a polynomial time TM M such that $L = L(M)$.

NP via TMs

Definition

L is an NP language iff there is a *non-deterministic* polynomial time TM M such that $L = L(M)$.

Non-deterministic TM: each step has a choice of moves

- $\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$.
  - Example: $\delta(q, a) = \{(q_1, b, L), (q_2, c, R), (q_3, a, R)\}$ means that M can non-deterministically choose one of the three possible moves from (q, a).
- $L(M)$: set of all strings s on which there *exists* some sequence of valid choices at each step that lead from q_0 to q_{accept}

Non-deterministic TMs vs certifiers

NP

Two definition of NP:

- L is in NP iff L has a polynomial time certifier $C(\cdot, \cdot)$.
- L is in NP iff L is decided by a non-deterministic polynomial time TM M.

Claim

Two definitions are equivalent.

Why?

Informal proof idea: the certificate t for C corresponds to non-deterministic choices of M and vice-versa.

In other words L is in NP iff L is accepted by a NTM which first guesses a proof t of length poly in input $|s|$ and then acts as a *deterministic* TM.

Non-determinism, guessing and verification

- A non-deterministic machine has choices at each step and accepts a string if there *exists* a set of choices which lead to a final state.
- Equivalently the choices can be thought of as *guessing* a solution and then *verifying* that solution. In this view all the choices are made a priori and hence the verification can be deterministic. The "guess" is the "proof" and the "verifier" is the "certifier".
- We reemphasize the asymmetry inherent in the definition of non-determinism. Strings in the language can be easily verified. No easy way to verify that a string is not in the language.

Algorithms: TMs vs RAM Model

Why do we use TMs some times and RAM Model other times?

- TMs are very simple: no complicated instruction set, no jumps/pointers, no explicit loops etc.
  - Simplicity is useful in proofs.
  - The "right" formal bare-bones model when dealing with subtleties.
- RAM model is a closer approximation to the running time/space usage of realistic computers for reasonable problem sizes
  - Not appropriate for certain kinds of formal proofs when algorithms can take super-polynomial time and space

"Hardest" Problems

Question

What is the hardest problem in NP? How do we define it?

Towards a definition

- Hardest problem must be in NP.
- Hardest problem must be at least as "difficult" as every other problem in NP.

NP-Complete Problems

Definition

A problem X is said to be **NP-Complete** if

- $X \in \mathrm{NP}$, and
- (Hardness) For any $Y \in \mathrm{NP}$, $Y \leq_P X$.

Solving NP-Complete Problems

Proposition

Suppose X is NP-COMPLETE. Then X can be solved in polynomial time if and only if $\mathrm{P} = \mathrm{NP}$.

Proof.

$\Rightarrow$ Suppose X can be solved in polynomial time

- Let $Y \in \mathrm{NP}$. We know $Y \leq_P X$.
- We showed that if $Y \leq_P X$ and X can be solved in polynomial time, then Y can be solved in polynomial time.
- Thus, every problem $Y \in \mathrm{NP}$ is such that $Y \in P$; $NP \subseteq P$.
- Since $\mathrm{P} \subseteq \mathrm{NP}$, we have $\mathrm{P} = \mathrm{NP}$.

$\Leftarrow$ Since $\mathrm{P} = \mathrm{NP}$, and $X \in \mathrm{NP}$, we have a polynomial time algorithm for X. $\square$

NP-Hard Problems

Definition

A problem X is said to be NP-Hard if

- (Hardness) For any $Y \in \mathrm{NP}$, $Y \leq_P X$

An NP-Hard problem need not be in NP!

Example: Halting problem is NP-Hard (why?) but not NP-Complete.

Consequences of proving NP-Completeness

If X is NP-Complete

- Since we believe $\mathrm{P} \neq \mathrm{NP}$,
- and solving X implies $\mathrm{P} = \mathrm{NP}$.

X is unlikely to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for X.
(This is proof by mob opinion — take with a grain of salt.)

NP-Complete Problems

Question

Are there any problems that are NP-COMPLETE?

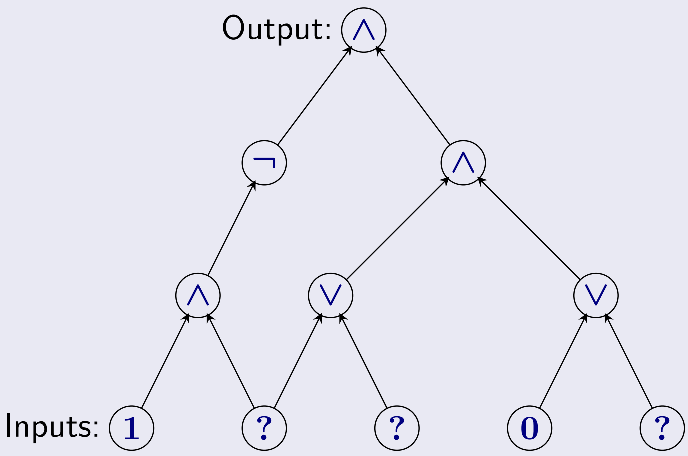
Answer

Yes! Many, many problems are NP-COMPLETE.

Circuits

Definition

A circuit is a directed *acyclic* graph with



- Input vertices (without incoming edges) labelled with **0**, **1** or a distinct variable
- Every other vertex is labelled $\vee$, $\wedge$ or $\neg$
- Single node output vertex with no outgoing edges

Cook-Levin Theorem

Definition (Circuit Satisfaction (CSAT).)

Given a circuit as input, is there an assignment to the input variables that causes the output to get value 1?

Theorem (Cook-Levin)

CSAT *is* NP-Complete.

Need to show

- **CSAT** is in NP
- *every* NP problem X reduces to **CSAT**.

CSAT: Circuit Satisfaction

Claim

CSAT *is in* NP.

- Certificate: Assignment to input variables.
- Certifier: Evaluate the value of each gate in a topological sort of DAG and check the output gate value.

CSAT is NP-hard: Idea

Need to show that *every* NP problem X reduces to **CSAT**.

What does it mean that $X \in \mathrm{NP}$?
$X \in \mathrm{NP}$ implies that there are polynomials $p()$ and $q()$ and certifier/verifier program C such that for every string s the following is true:

- If s is a YES instance ($s \in X$) then there is a *proof* t of length $p(|s|)$ such that $C(s, t)$ says YES.
- If s is a NO instance ($s \notin X$) then for every string t of length at $p(|s|)$, $C(s, t)$ says NO.
- $C(s, t)$ runs in time $q(|s| + |t|)$ time (hence polynomial time).

Reducing X to CSAT

X is in NP means we have access to $p(), q(), C(\cdot, \cdot)$.
What is $C(\cdot, \cdot)$? It is a program or equivalently a Turing Machine!
How are $p()$ and $q()$ given? As numbers.
Example: if 3 is given then $p(n) = n^3$.

Thus an NP problem is essentially a three tuple $< p, q, C >$ where C is either a program or a TM.

Reducing X to CSAT

Thus an NP problem is essentially a three tuple $< p, q, C >$ where C is either a program or TM.

Problem X: Given string s, is $s \in X$?

Same as the following: is there a proof t of length $p(|s|)$ such that $C(s, t)$ says YES.

How do we reduce X to **CSAT**? Need an algorithm $\mathcal{A}$ that

- takes s (and $< p, q, C >$) and creates a circuit G in polynomial time in $|s|$ (note that $< p, q, C >$ are fixed).
- G is satisfiable if and only if there is a proof t such that $C(s, t)$ says YES.

Reducing X to CSAT

How do we reduce X to **CSAT**? Need an algorithm $\mathcal{A}$ that

- takes s (and $< p, q, C >$) and creates a circuit G in polynomial time in $|s|$ (note that $< p, q, C >$ are fixed).
- G is satisfiable if and only if there is a proof t such that $C(s, t)$ says YES

Simple but Big Idea: Programs are essentially the same as Circuits!

- Convert $C(s, t)$ into a circuit G with t as unknown inputs (rest is known including s)
- We know that $|t| = p(|s|)$ so express boolean string t as $p(|s|)$ variables $t_1, t_2, \ldots, t_k$ where $k = p(|s|)$.
- Asking if there is a proof t that makes $C(s, t)$ say YES is same as whether there is an assignment of values to "unknown" variables $t_1, t_2, \ldots, t_k$ that will make G evaluate to true/YES.

Example: Independent Set

- Problem: Does $G = (V, E)$ have an **Independent Set** of size $\geq k$?
  - Certificate: Set $S \subseteq V$
  - Certifier: Check $|S| \geq k$ and no pair of vertices in S is connected by an edge

Formally, why is **Independent Set** in NP?

Example: Independent Set

Formally why is **Independent Set** in NP?

- Input: $< n, y_{1,1}, y_{1,2}, \ldots, y_{1,n}, y_{2,1}, \ldots, y_{2,n}, \ldots, y_{n,1}, \ldots, y_{n,n}, k >$ encodes $< G, k >$.
  - n is number of vertices in G
  - $y_{i,j}$ is a bit which is 1 if edge (i, j) is in G and 0 otherwise (adjacency matrix representation)
  - k is size of independent set.
- Certificate: $t = t_1 t_2 \ldots t_n$. Interpretation is that t_i is 1 if vertex i is in the independent set, 0 otherwise.

Certifier for **Independent Set**

Certifier $C(s, t)$ for **Independent Set**:

if $(t_1 + t_2 + \ldots + t_n < k)$ **then**
  return NO
else
  for each (i, j) **do**
    if $(t_i \wedge t_j \wedge y_{i,j})$ **then**
      return NO

return YES

Example: Independent Set

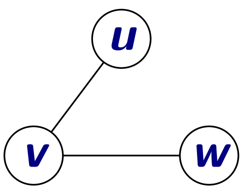
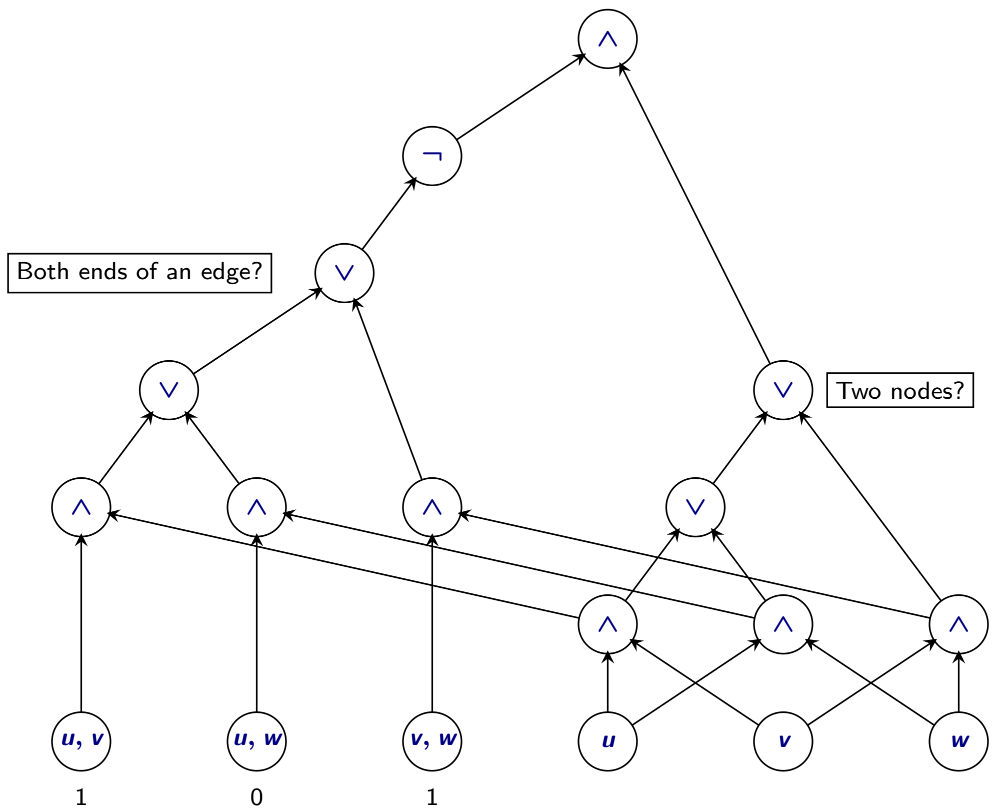


Figure: Graph G with $k = 2$



Circuit from Certifier

Programs, Turing Machines and Circuits

Consider "program" A that takes $f(|s|)$ steps on input string s.

Question: What computer is the program running on and what does *step* mean?
Real computers difficult to reason with mathematically because

- instruction set is too rich
- pointers and control flow jumps in one step
- assumption that pointer to code fits in one word

Turing Machines

- simpler model of computation to reason with
- can simulate real computers with *polynomial* slow down
- all moves are *local* (head moves only one cell)

Certifiers that at TMs

Assume $C(\cdot, \cdot)$ is a (deterministic) Turing Machine M

Problem: Given M, input s, p, q decide if there is a proof t of length $p(|s|)$ such that M on s, t will halt in $q(|s|)$ time and say YES.

There is an algorithm $\mathcal{A}$ that can reduce above problem to **CSAT** mechanically as follows.

- $\mathcal{A}$ first computes $p(|s|)$ and $q(|s|)$.
- Knows that M can use at most $q(|s|)$ memory/tape cells
- Knows that M can run for at most $q(|s|)$ time
- Simulates the evolution of the state of M and memory over time using a big circuit.

Simulation of Computation via Circuit

- Think of M's state at time ℓ as a string $x^\ell = x_1 x_2 \ldots x_k$ where each $x_i \in \{0, 1, B\} \times Q \cup \{q_{-1}\}$.
- At time 0 the state of M consists of input string s a guess t (unknown variables) of length $p(|s|)$ and rest $q(|s|)$ blank symbols.
- At time $q(|s|)$ we wish to know if M stops in q_{accept} with say all blanks on the tape.
- We write a circuit C_ℓ which captures the transition of M from time ℓ to time $\ell + 1$.
- Composition of the circuits for all times 0 to $q(|s|)$ gives a big (still poly) sized circuit $\mathcal{C}$
- The final output of $\mathcal{C}$ should be true if and only if the entire state of M at the end leads to an accept state.

NP-Hardness of Circuit Satisfaction

Key Ideas in reduction:

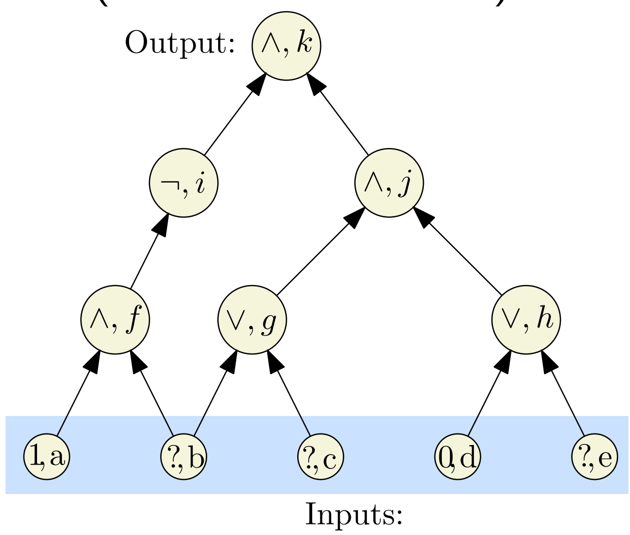
- Use TMs as the code for certifier for simplicity
- Since $p()$ and $q()$ are known to $\mathcal{A}$, it can set up all required memory and time steps in advance
- Simulate computation of the TM from one time to the next as a circuit that only looks at three adjacent cells at a time

Note: Above reduction can be done to **SAT** as well. Reduction to **SAT** was the original proof of Steve Cook.

SAT is NP-Complete

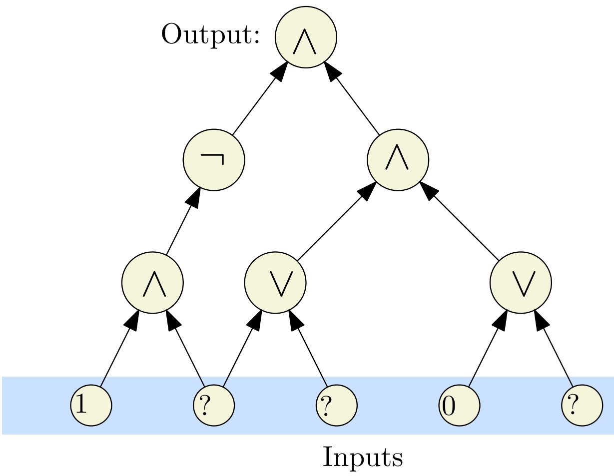
- We have seen that **SAT** $\in$ NP
- To show NP-Hardness, we will reduce Circuit Satisfiability (**CSAT**) to **SAT**

  Instance of **CSAT** (we label each node):

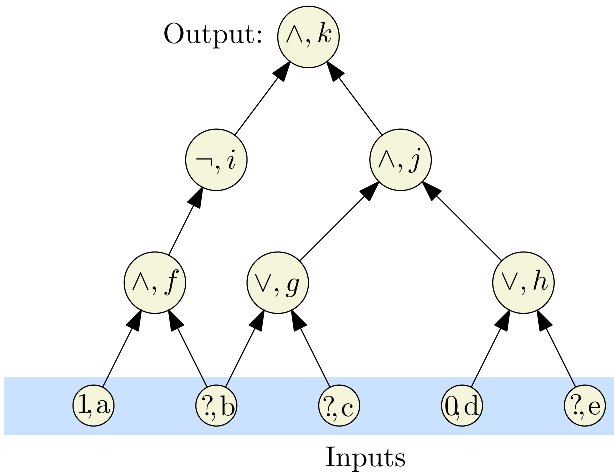


Converting a circuit into a CNF formula

Label the nodes



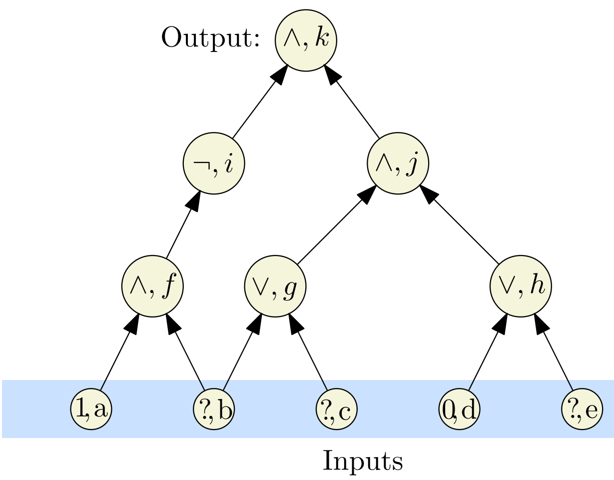
(A) Input circuit



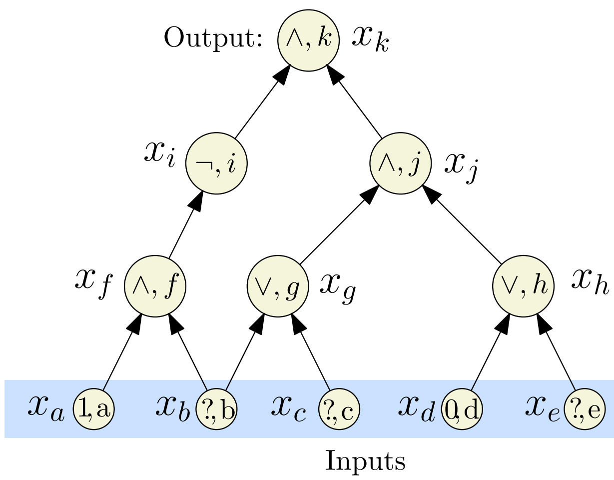
(B) Label the nodes.

Converting a circuit into a CNF formula

Introduce a variable for each node



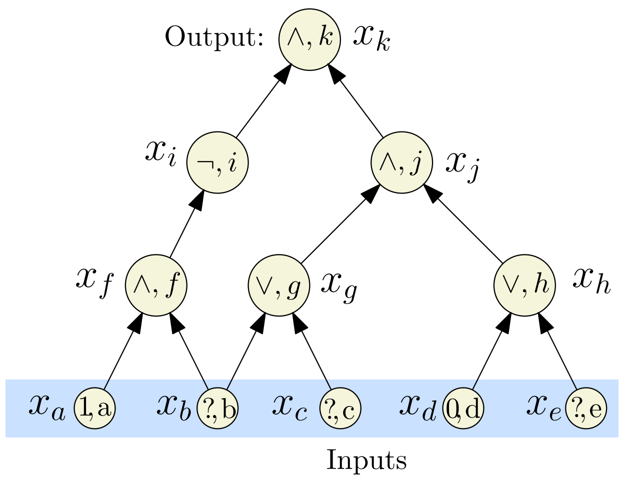
(B) Label the nodes.



(C) Introduce var for each node.

Converting a circuit into a CNF formula

Write a sub-formula for each variable that is true if the var is computed correctly.



(C) Introduce var for each node.

$\mathbf{x_k}$ (Demand a sat' assignment!)

$\mathbf{x_k = x_i \wedge x_k}$

$\mathbf{x_j = x_g \wedge x_h}$

$\mathbf{x_i = \neg x_f}$

$\mathbf{x_h = x_d \vee x_e}$

$\mathbf{x_g = x_b \vee x_c}$

$\mathbf{x_f = x_a \wedge x_b}$

$\mathbf{x_d = 0}$

$\mathbf{x_a = 1}$

(D) Write a sub-formula for each variable that is true if the var is computed correctly.

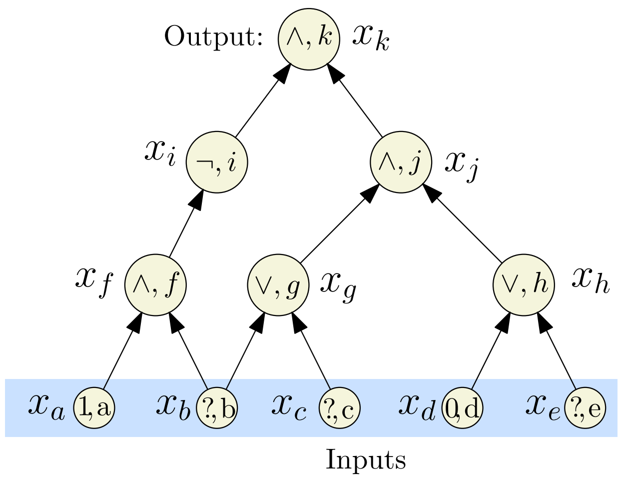
Converting a circuit into a CNF formula

Convert each sub-formula to an equivalent CNF formula

$\mathbf{x_k}$	$\mathbf{x_k}$
$\mathbf{x_k = x_i \wedge x_j}$	$\mathbf{(\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \wedge (x_k \vee \neg x_i \vee \neg x_j)}$
$\mathbf{x_j = x_g \wedge x_h}$	$\mathbf{(\neg x_j \vee x_g) \wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h) \wedge}$
$\mathbf{x_i = \neg x_f}$	$\mathbf{(x_i \vee x_f) \wedge (\neg x_i \vee x_f) \wedge}$
$\mathbf{x_h = x_d \vee x_e}$	$\mathbf{(x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \wedge (\neg x_h \vee x_d \vee x_e)}$
$\mathbf{x_g = x_b \vee x_c}$	$\mathbf{(x_g \vee \neg x_b) \wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c)}$
$\mathbf{x_f = x_a \wedge x_b}$	$\mathbf{(\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \wedge (x_f \vee \neg x_a \vee \neg x_b)}$
$\mathbf{x_d = 0}$	$\mathbf{\neg x_d}$
$\mathbf{x_a = 1}$	$\mathbf{x_a}$

Converting a circuit into a CNF formula

Take the conjunction of all the CNF sub-formulas



$$x_k \wedge (\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \wedge (x_k \vee \neg x_i \vee \neg x_j) \wedge (\neg x_j \vee x_g) \wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h) \wedge (x_i \vee x_f) \wedge (\neg x_i \vee x_f) \wedge (x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \wedge (\neg x_h \vee x_d \vee x_e) \wedge (x_g \vee \neg x_b) \wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c) \wedge (\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \wedge (x_f \vee \neg x_a \vee \neg x_b) \wedge (\neg x_d) \wedge x_a$$

We got a CNF formula that is satisfiable if and only if the original circuit is satisfiable.

Reduction: CSAT $\leq_P$ SAT

- For each gate (vertex) v in the circuit, create a variable x_v
- Case $\neg$: v is labeled $\neg$ and has one incoming edge from u (so $x_v = \neg x_u$). In SAT formula generate, add clauses $(x_u \vee x_v)$, $(\neg x_u \vee \neg x_v)$. Observe that

$$x_v = \neg x_u \text{ is true} \iff \begin{matrix}(x_u \vee x_v)\\(\neg x_u \vee \neg x_v)\end{matrix} \text{ both true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

- Case $\vee$: So $x_v = x_u \vee x_w$. In **SAT** formula generated, add clauses $(x_v \vee \neg x_u)$, $(x_v \vee \neg x_w)$, and $(\neg x_v \vee x_u \vee x_w)$. Again, observe that

$$x_v = x_u \vee x_w \text{ is true} \iff \begin{matrix} (x_v \vee \neg x_u), \\ (x_v \vee \neg x_w), \\ (\neg x_v \vee x_u \vee x_w) \end{matrix} \quad \text{all true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

- Case $\wedge$: So $x_v = x_u \wedge x_w$. In **SAT** formula generated, add clauses $(\neg x_v \vee x_u)$, $(\neg x_v \vee x_w)$, and $(x_v \vee \neg x_u \vee \neg x_w)$. Again observe that

$$x_v = x_u \wedge x_w \text{ is true} \iff \begin{matrix} (\neg x_v \vee x_u), \\ (\neg x_v \vee x_w), \\ (x_v \vee \neg x_u \vee \neg x_w) \end{matrix} \quad \text{all true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

- If v is an input gate with a fixed value then we do the following. If $x_v = 1$ add clause x_v. If $x_v = 0$ add clause $\neg x_v$
- Add the clause x_v where v is the variable for the output gate

Correctness of Reduction

Need to show circuit C is satisfiable iff φ_C is satisfiable

$\Rightarrow$ Consider a satisfying assignment a for C
  - Find values of all gates in C under a
  - Give value of gate v to variable x_v; call this assignment a'
  - a' satisfies φ_C (exercise)

$\Leftarrow$ Consider a satisfying assignment a for φ_C
  - Let a' be the restriction of a to only the input variables
  - Value of gate v under a' is the same as value of x_v in a
  - Thus, a' satisfies C

Theorem

SAT is NP-COMPLETE.

Proving that a problem X is NP-Complete

To prove X is NP-COMPLETE, show

- Show X is in NP.
  - certificate/proof of polynomial size in input
  - polynomial time certifier $C(s, t)$
- Reduction from a known NP-COMPLETE problem such as **CSAT** or **SAT** to X

SAT $\leq_P$ X implies that every NP problem $Y \leq_P X$. Why? Transitivity of reductions:

$Y \leq_P SAT$ and $SAT \leq_P X$ and hence $Y \leq_P X$.

NP-Completeness via Reductions

- **CSAT** is NP-COMPLETE.
- **CSAT** $\leq_P$ **SAT** and **SAT** is in NP and hence SAT is NP-COMPLETE.
- **SAT** $\leq_P$ **3-SAT** and hence 3-SAT is NP-COMPLETE.
- **3-SAT** $\leq_P$ Independent Set (which is in NP) and hence **Independent Set** is NP-COMPLETE.
- **Vertex Cover** is NP-COMPLETE.
- **Clique** is NP-COMPLETE.

Hundreds and thousands of different problems from many areas of science and engineering have been shown to be NP-COMPLETE.

A surprisingly frequent phenomenon!