More Network Flow Applications

Lecture 19
November 3, 2011
Part I

Baseball Pennant Race
49ers, Young Get Big Break

Quarterback may return

By Gary Scan
Chronicle Staff Writer

The bye week has come at a perfect time for the 49ers and quarterback Steve Young. If they had a game next Sunday, there's a good chance Young would not play.

Just the needed extra muscle on his up-

Giants Officially Leave the NL West Race

By Nancy Gay
Chronicle Staff Writer

With the smack of another National League West rival 500 miles away, the Giants' run at the division tit-

CARDINALS 6
GIANTS 2

In San Diego, Greg Vaughn's three-run homer in the eighth pushed the Padres over the Pirates and officially shoved the rest of the Giants' season into the back-

remaining 23 games before an announced crowd of 10,207 at Candlestick Park, the Giants fell 19
games off the lead.

As it is, the worst the Padres (80-65) can finish is 80-82. The Giants have fallen to 59-83 with 20
games left; they cannot win 80 games. Coming off a miserable 3-8 mark on a three-city road trip that saw their road record drop to 27-
47, the Giants were hoping to get off on the right foot in their longest homestand of the year (15 games, 14 days).

Financing in Place For Giants' New Stadium
SEE PAGE 81, MAIN NEWS

“Where we are, you're going to be eliminated sooner or later,”
Baker said quietly. “But it doesn’t alter the fact that we've still got to play ball. You've still got to play hard, the fans come out to watch you play. You've got to play for the fact of loving to play, no matter where you are in the standings.

“Where we are, you're going to be eliminated sooner or later,”
Baker said quietly. “But it doesn’t alter the fact that we've still got to play ball. You've still got to play hard, the fans come out to watch you play. You've got to play for the fact of loving to play, no matter where you are in the standings.
Pennant Race: Example

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Can Boston win the pennant?

No, because Boston can win at most 91 games.
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Abstracting the Problem

Given

- A set of teams $S$
- For each $x \in S$, the current number of wins $w_x$
- For any $x, y \in S$, the number of remaining games $g_{xy}$ between $x$ and $y$
- A team $z$

Can $z$ win the pennant?
Towards a Reduction

\( \bar{z} \) can win the pennant if

- \( \bar{z} \) wins at least \( m \) games
- no other team wins more than \( m \) games
Towards a Reduction

\( \overline{z} \) can win the pennant if

- \( \overline{z} \) wins at least \( m \) games
  - to maximize \( \overline{z} \)'s chances we make \( \overline{z} \) win all its remaining games
    and hence \( m = w_{\overline{z}} + \sum_{x \in S} g_{x\overline{z}} \)
  - no other team wins more than \( m \) games
Towards a Reduction

\( \overline{z} \) can win the pennant if

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  - no other team wins more than \( m \) games
    - for each \( x, y \in S \) the \( g_{xy} \) games between them have to be assigned to either \( x \) or \( y \).
    - each team \( x \neq \overline{z} \) can win at most \( m - w_x - g_{x \overline{z}} \) remaining games

Is there an assignment of remaining games to teams such that no team \( x \neq \overline{z} \) wins more than \( m - w_x \) games?
Flow Network: The basic gadget

- \( s \): source
- \( t \): sink
- \( x, y \): two teams
- \( g_{xy} \): number of games remaining between \( x \) and \( y \).
- \( w_x \): number of points \( x \) has.
- \( m \): maximum number of points \( x \) can win before team of interest is eliminated.

\[
\begin{align*}
\text{Flow Network: The basic gadget} \\
\text{Diagram showing nodes and edges with labels.}
\end{align*}
\]
Flow Network: An Example

Can Boston win?

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<td>90</td>
<td>11</td>
<td>—</td>
<td>1</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Baltimore</td>
<td>88</td>
<td>6</td>
<td>1</td>
<td>—</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Toronto</td>
<td>87</td>
<td>11</td>
<td>6</td>
<td>1</td>
<td>—</td>
<td>4</td>
</tr>
<tr>
<td>Boston</td>
<td>79</td>
<td>12</td>
<td>4</td>
<td>4</td>
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• $m = 79 + 12 = 91$: Boston can get at most 91 points.
Notations

- $S$: set of teams,
- $w_x$: wins for each team, and
- $g_{xy}$: games left between $x$ and $y$.
- $m$: be the maximum number of wins for $\bar{z}$, and $S' = S \setminus \{\bar{z}\}$.

Reduction

Construct the flow network $G$ as follows:

- One vertex $v_x$ for each team $x \in S'$, one vertex $u_{xy}$ for each pair of teams $x$ and $y$ in $S'$
- A new source vertex $s$ and sink $t$
- Edges $(u_{xy}, v_x)$ and $(u_{xy}, v_y)$ of capacity $\infty$
- Edges $(s, u_{xy})$ of capacity $g_{xy}$
- Edges $(v_x, t)$ of capacity equal $m - w_x$
Correctness of reduction

Theorem

$G'$ has a maximum flow of value $g^* = \sum_{x, y \in S'} g_{xy}$ if and only if $\bar{z}$ can win the most number of games (including possibly a tie with other teams).
Proof of Correctness

Proof.

Existence of $g^*$ flow $\Rightarrow \bar{z}$ wins pennant

- An integral flow saturating edges out of $s$, ensures that each remaining game between $x$ and $y$ is added to win total of either $x$ or $y$
- Capacity on $(v_x, t)$ edges ensures that no team wins more than $m$ games

Conversely, $\bar{z}$ wins pennant $\Rightarrow$ flow of value $g^*$

- Scenario determines flow on edges; if $x$ wins $k$ of the games against $y$, then flow on $(u_{xy}, v_x)$ edge is $k$ and on $(u_{xy}, v_y)$ edge is $g_{xy} - k$
Proof that $\bar{z}$ cannot with the pennant

- Suppose $\bar{z}$ cannot win the pennant since $g^* < g$. How do we prove to some one *compactly* that $\bar{z}$ cannot win the pennant?
- Show them the min-cut in the reduction flow network!
- See text book for a natural interpretation of the min-cut as a certificate.
Proof that $\overline{z}$ cannot with the pennant

- Suppose $\overline{z}$ cannot win the pennant since $g^* < g$. How do we prove to some one compactly that $\overline{z}$ cannot win the pennant?
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Part II

An Application of Min-Cut to Project Scheduling
Project Scheduling

Problem:

- \( n \) projects/tasks \( 1, 2, \ldots, n \)
- dependencies between projects: \( i \) depends on \( j \) implies \( i \) cannot be done unless \( j \) is done. dependency graph is acyclic
- each project \( i \) has a cost/profit \( p_i \)
  - \( p_i < 0 \) implies \( i \) requires a cost of \(-p_i\) units
  - \( p_i > 0 \) implies that \( i \) generates \( p_i \) profit

Goal: Find projects to do so as to maximize profit.
Example

\[\begin{align*}
-2 \quad & \quad \infty \\
\infty \quad & \quad -3 \\
\infty \quad & \quad \infty \\
-3 \quad & \quad -5 \\
\infty \quad & \quad -8
\end{align*}\]
Notation

For a set $A$ of projects:

- $A$ is a valid solution if $A$ is dependency closed, that is for every $i \in A$, all projects that $i$ depends on are also in $A$
- $\text{profit}(A) = \sum_{i \in A} p_i$. Can be negative or positive

Goal: find valid $A$ to maximize $\text{profit}(A)$
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**Goal:** find valid $A$ to maximize $\text{profit}(A)$.
Idea: Reduction to Minimum-Cut

Finding a set of projects is partitioning the projects into two sets: those that are done and those that are not done.

Can we express this is a minimum cut problem?

Several issues:

- We are interested in maximizing profit but we can solve minimum cuts
- We need to convert negative profits into positive capacities
- Need to ensure that chosen projects is a valid set
- The cut value captures the profit of the chosen set of projects
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Reduction to Minimum-Cut

Note: We are reducing a *maximization* problem to a *minimization* problem.

- projects represented as nodes in a graph
- if \( i \) depends on \( j \) then \((i, j)\) is an edge
- add source \( s \) and sink \( t \)
- for each \( i \) with \( p_i > 0 \) add edge \((s, i)\) with capacity \( p_i \)
- for each \( i \) with \( p_i < 0 \) add edge \((i, t)\) with capacity \(-p_i\)
- for each dependency edge \((i, j)\) put capacity \(\infty\) (more on this later)
Reduction: Flow Network Example

![Flow Network Example Diagram]
Algorithm:
- form graph as in previous slide
- compute $s$-$t$ minimum cut $(A, B)$
- output the projects in $A - \{s\}$
Understanding the Reduction

Let \( C = \sum_{i: p_i > 0} p_i \): maximum possible profit.

Observation: The minimum \( s-t \) cut value is \( \leq C \). Why?

Lemma

Suppose \((A, B)\) is an \( s-t \) cut of finite capacity (no \( \infty \)) edges. Then projects in \( A - \{s\} \) are a valid solution.

Proof.

If \( A - \{s\} \) is not a valid solution then there is a project \( i \in A \) and a project \( j \notin A \) such that \( i \) depends on \( j \).

Since \((i, j)\) capacity is \( \infty \), implies \((A, B)\) capacity is \( \infty \), contradicting assumption.
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Not a valid set of projects
Example
Correctness of Reduction

Recall that for a set of projects $X$, $\text{profit}(X) = \sum_{i \in X} p_i$.

Lemma

Suppose $(A, B)$ is an $s$-$t$ cut of finite capacity (no $\infty$) edges. Then $c(A, B) = C - \text{profit}(A - \{s\})$.

Proof.

Edges in $(A, B)$:

- $(s, i)$ for $i \in B$ and $p_i > 0$: capacity is $p_i$
- $(i, t)$ for $i \in A$ and $p_i < 0$: capacity is $-p_i$
- cannot have $\infty$ edges
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Edges in $(A, B)$:

- $(s, i)$ for $i \in B$ and $p_i > 0$: capacity is $p_i$
- $(i, t)$ for $i \in A$ and $p_i < 0$: capacity is $-p_i$
- cannot have $\infty$ edges
Proof contd

For project set $A$ let

- $\text{cost}(A) = \sum_{i \in A : p_i < 0} -p_i$
- $\text{benefit}(A) = \sum_{i \in A : p_i > 0} p_i$
- $\text{profit}(A) = \text{benefit}(A) - \text{cost}(A)$.

Proof.

\[
c(A, B) = \text{cost}(A) + \text{benefit}(B) \\
= \text{cost}(A) - \text{benefit}(A) + \text{benefit}(A) + \text{benefit}(B) \\
= -\text{profit}(A) + C \\
= C - \text{profit}(A)
\]
We have shown that if \((A, B)\) is an \(s-t\) cut in \(G\) with finite capacity then

- \(A - \{s\}\) is a valid set of projects
- \(c(A, B) = C - \text{profit}(A - \{s\})\)

Therefore a minimum \(s-t\) cut \((A^*, B^*)\) gives a maximum profit set of projects \(A^* - \{s\}\) since \(C\) is fixed.

**Question:** How can we use \(\infty\) in a real algorithm?

Set capacity of \(\infty\) arcs to \(C + 1\) instead. Why does this work?
Correctness of Reduction contd

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Part III

Extensions to Maximum-Flow Problem
Lower Bounds and Costs

Two generalizations:

- flow satisfies $f(e) \leq c(e)$ for all $e$. suppose we are given lower bounds $\ell(e)$ for each $e$. can we find a flow such that $\ell(e) \leq f(e) \leq c(e)$ for all $e$?

- suppose we are given a cost $w(e)$ for each edge. cost of routing flow $f(e)$ on edge $e$ is $w(e)f(e)$. can we (efficiently) find a flow (of at least some given quantity) at minimum cost?

Many applications.
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Flows with Lower Bounds

Definition

A flow in a network \( G = (V, E) \), is a function \( f : E \to \mathbb{R}^{\geq 0} \) such that

- **Capacity Constraint:** For each edge \( e \), \( f(e) \leq c(e) \)
- **Lower Bound Constraint:** For each edge \( e \), \( f(e) \geq \ell(e) \)
- **Conservation Constraint:** For each vertex \( v \)

\[
\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)
\]

Question: Given \( G \) and \( c(e) \) and \( \ell(e) \) for each \( e \), is there a flow? As difficult as finding an \( s-t \) maximum-flow without lower bounds!
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Regular flow via lower bounds

Given usual flow network $G$ with source $s$ and sink $t$, create lower-bound flow network $G'$ as follows:

- set $\ell(e) = 0$ for each $e$ in $G$
- add new edge $(t, s)$ with lower bound $\nu$ and upper bound $\infty$

Claim: there exists a flow of value $\nu$ from $s$ to $t$ in $G$ if and only if there exists a feasible flow with lower bounds in $G'$

Above reduction show that lower bounds on flows are naturally related to circulations. With lower bounds, cannot guarantee acyclic flows from $s$ to $t$. 
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Flows with Lower Bounds

- Flows with lower bounds can be reduced to standard maximum flow problem. See text book. Reduction goes via circulations.
- If all bounds are integers then there is a flow that is integral. Useful in applications.
Survey Design: Application of Flows with Lower Bounds

- Design survey to find information about $n_1$ products from $n_2$ customers
- Can ask customer questions only about products purchased in the past
- Customer can only be asked about at most $c'_i$ products and at least $c_i$ products
- For each product need to ask at least $p_i$ consumers and at most $p'_i$ consumers
Reduction to Circulation

- include edge \((i, j)\) is customer \(i\) has bought product \(j\)
- Add edge \((t, s)\) with lower bound 0 and upper bound \(\infty\).
  - Consumer \(i\) is asked about product \(j\) if the integral flow on edge \((i, j)\) is 1
Minimum Cost Flows

- **Input:** Given a flow network $G$ and also edge costs, $w(e)$ for edge $e$, and a flow requirement $F$.
- **Goal:** Find a *minimum cost* flow of value $F$ from $s$ to $t$.

Given flow $f : E \rightarrow R^+$, cost of flow $= \sum_{e \in E} w(e)f(e)$.
Minimum Cost Flow: Facts

- Problem can be solved efficiently in polynomial time
  - $O(nm \log C \log(nW))$ time algorithm where $C$ is maximum edge capacity and $W$ is maximum edge cost
  - $O(m \log n(m + n \log n))$ time strongly polynomial time algorithm

- For integer capacities there is always an optimum solutions in which flow is integral
How much damage can a single path cause?

Consider the following network. All the edges have capacity $1$. Clearly the maximum flow in this network has value $4$.

Why removing the shortest path might ruin everything

- However... The shortest path between $s$ and $t$ is the blue path.
- And if we remove the shortest path, $s$ and $t$ become disconnected, and the maximum flow drop to $0$. 