Applications of Network Flows

Lecture 18

November 1, 2011

Network Flow: Facts to Remember

Flow network: directed graph G, capacities c, source s, sink t

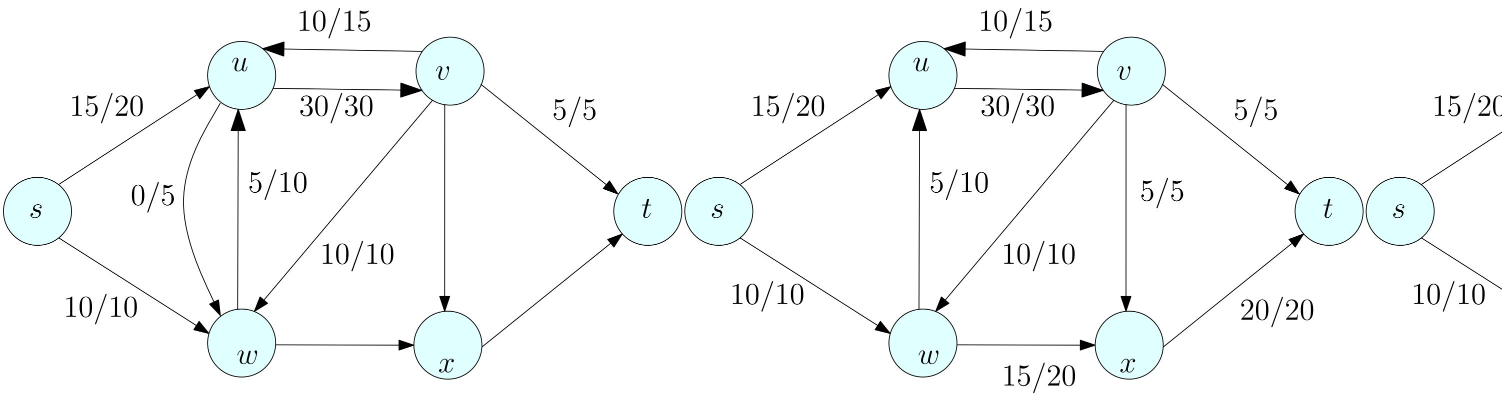
- Maximum s-t flow can be computed:
  - Using Ford-Fulkerson algorithm in $O(mC)$ time when capacities are integral and C is an upper bound on the flow
  - Using variant of algorithm in $O(m^2 \log C)$ time when capacities are integral
  - Using Edmonds-Karp algorithm in $O(m^2 n)$ time when capacities are rational (strongly polynomial time algorithm).
- If capacities are integral then there is a maximum flow that is integral and above algorithms give an integral max flow.
- Given a flow of value v, can decompose into $O(m + n)$ flow paths of same total value v. integral flow implies integral flow on paths.
- Maximum flow is equal to the minimum cut and minimum cut can be found in $O(m + n)$ time given any maximum flow.

Paths, Cycles and Acyclicity of Flows

Definition

Given a flow network $G = (V, E)$ and a flow $f : E \to \mathbb{R}^{\geq 0}$ on the edges, the support of f is the set of edges $E' \subseteq E$ with non-zero flow on them. That is, $E' = \{e \in E \mid f(e) > 0\}$.

Question: Given a flow f, can there by cycles in its support?



Acyclicity of Flows

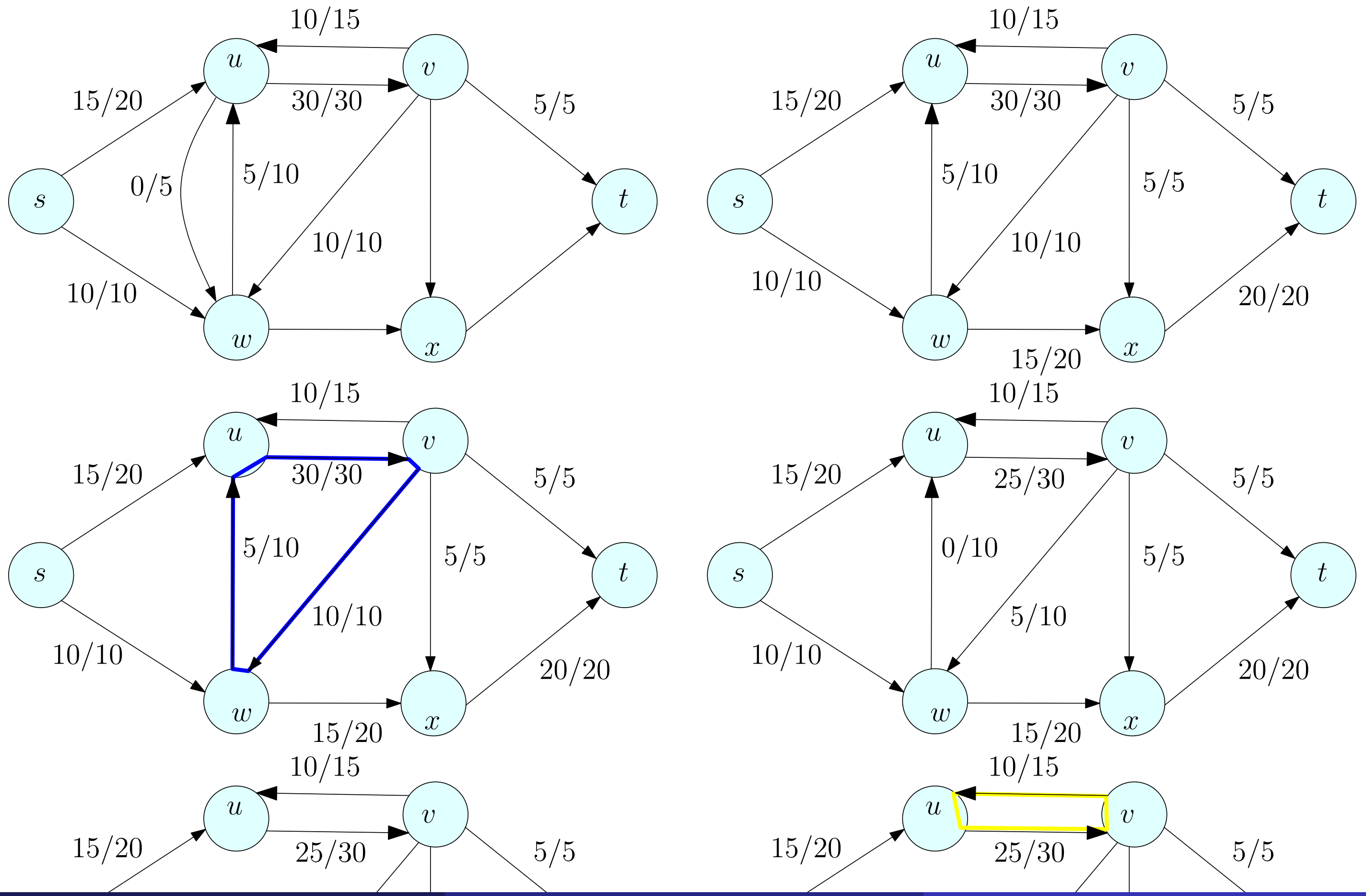
Proposition

In any flow network, if f is a flow then there is another flow f' such that the support of f' is an acyclic graph and $v(f') = v(f)$. Further if f is an integral flow then so is f'.

Proof.

- $E' = \{e \in E \mid f(e) > 0\}$, support of f.
- Suppose there is a directed cycle C in E'
- Let e' be the edge in C with least amount of flow
- For each $e \in C$, reduce flow by $f(e')$. Remains a flow. Why?
- flow on e' is reduced to 0
- Claim: Flow value from s to t does not change. Why?
- Iterate until no cycles □

Example



Flow Decomposition

Lemma

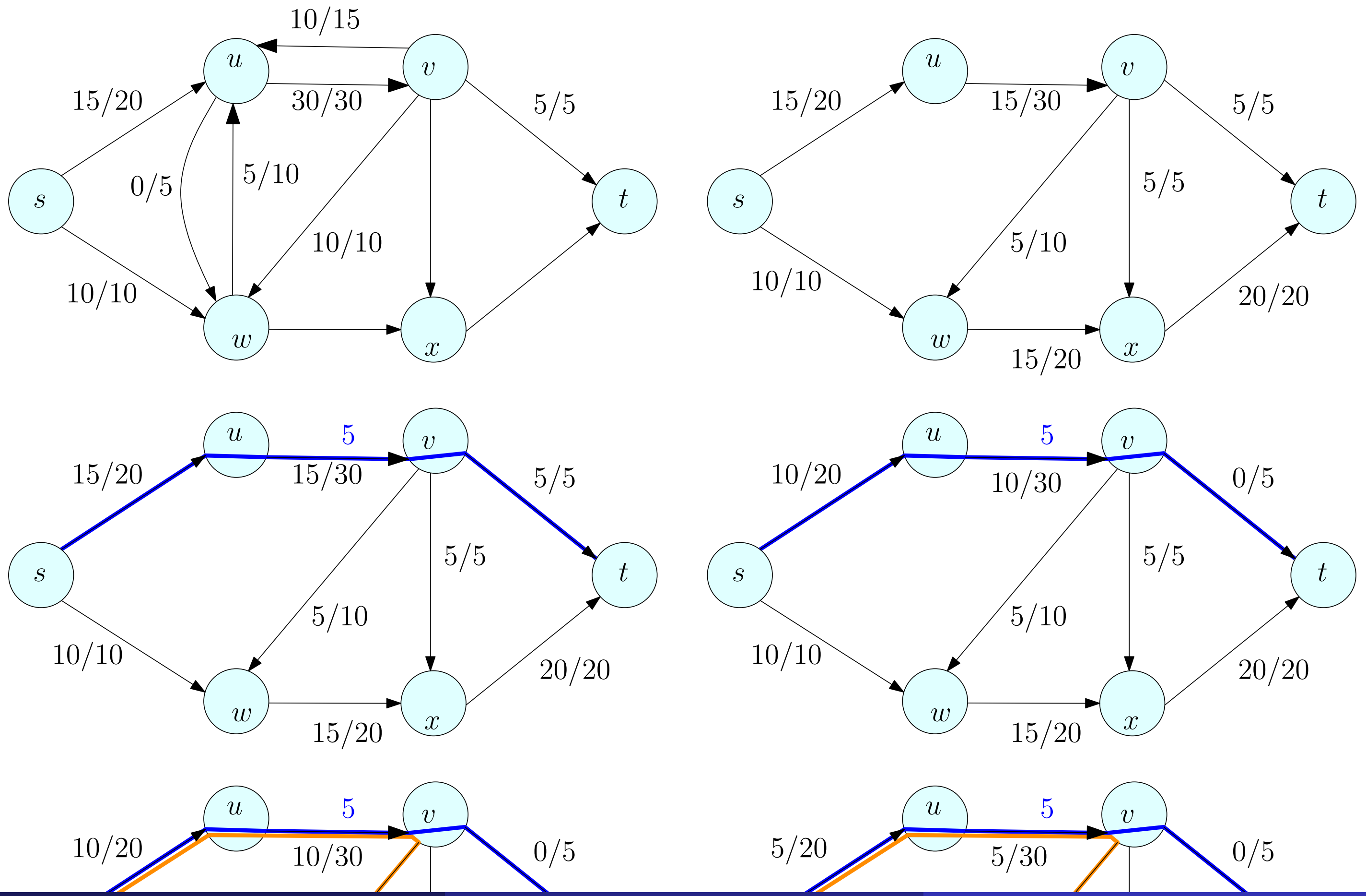
Given an edge based flow $\boldsymbol{f} : \boldsymbol{E} \rightarrow \mathbb{R}^{\geq 0}$, *there exists a collection of paths* $\mathcal{P}$ *and cycles* $\mathcal{C}$ *and an assignment of flow to them* $\boldsymbol{f'} : \mathcal{P} \cup \mathcal{C} \rightarrow \mathbb{R}^{\geq 0}$ *such that:*

- $|\mathcal{P} \cup \mathcal{C}| \leq \boldsymbol{m}$
- *for each* $\boldsymbol{e} \in \boldsymbol{E}$, $\sum_{P \in \mathcal{P}: e \in P} \boldsymbol{f'}(\boldsymbol{P}) + \sum_{C \in \mathcal{C}: e \in C} \boldsymbol{f'}(\boldsymbol{C}) = \boldsymbol{f}(\boldsymbol{e})$
- $\boldsymbol{v}(\boldsymbol{f}) = \sum_{P \in \mathcal{P}} \boldsymbol{f'}(\boldsymbol{P})$.
- *if* $\boldsymbol{f}$ *is integral then so are* $\boldsymbol{f'}(\boldsymbol{P})$ *and* $\boldsymbol{f'}(\boldsymbol{C})$ *for all* $\boldsymbol{P}$ *and* $\boldsymbol{C}$

Proof Idea.

- Remove all cycles as in previous proposition.
- Next, decompose into paths as in previous lecture.
- Exercise: verify claims. $\square$

Example



Flow Decomposition

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- $|\mathcal{P} \cup \mathcal{C}| \leq m$
- *for each* $e \in E$, $\sum_{P \in \mathcal{P} : e \in P} f'(P) + \sum_{C \in \mathcal{C} : e \in C} f'(C) = f(e)$
- $v(f) = \sum_{P \in \mathcal{P}} f'(P)$.
- *if* f *is integral then so are* $f'(P)$ *and* $f'(C)$ *for all* P *and* C

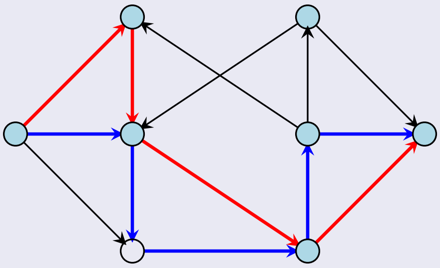
Above flow decomposition can be computed in $O(m^2)$ *time.*

Part I

Network Flow Applications I

Edge-Disjoint Paths in Directed Graphs

Definition



A set of paths is edge disjoint if no two paths share an edge.

Problem

Given a directed graph with two special vertices s and t, find the *maximum* number of edge disjoint paths from s to t

Applications: Fault tolerance in routing — edges/nodes in networks can fail. Disjoint paths allow for planning backup routes in case of failures.

Reduction to Max-Flow

Problem

Given a directed graph G with two special vertices s and t, find the maximum number of edge disjoint paths from s to t.

Reduction

Consider G as a flow network with edge capacities 1, and find max-flow.

Correctness of Reduction

Lemma

If G has k edge disjoint paths $P_1, P_2, \ldots, P_k$ then there is an s-t flow of value k.

Proof.

Set $f(e) = 1$ if e belongs to one of the paths $P_1, P_2, \ldots, P_k$; other-wise set $f(e) = 0$. This defines a flow of value k. ☐

Correctness of Reduction

Lemma

If G has a flow of value k then there are k edge disjoint paths between s and t.

Proof.

- Capacities are all 1 and hence there is integer flow of value k, that is $f(e) = 0$ or $f(e) = 1$ for each e.
- Decompose flow into paths of same value
- Flow on each path is either 1 or 0
- Hence there are k paths $P_1, P_2, \ldots, P_k$ with flow of 1 each
- Paths are edge-disjoint since capacities are 1.

☐

Running Time

Theorem

The number of edge disjoint paths in G can be found in $O(mn)$ time.

Run Ford-Fulkerson algorithm. Maximum possible flow is n and hence run-time is $O(nm)$.

Menger's Theorem

Theorem (Menger)

Let G be a directed graph. The minimum number of edges whose removal disconnects s from t (the minimum-cut between s and t) is equal to the maximum number of edge-disjoint paths in G between s and t.

Proof.

Maxflow-mincut theorem and integrality of flow. □

Menger proved his theorem before Maxflow-Mincut theorem!
Maxflow-Mincut theorem is a generalization of Menger's theorem to capacitated graphs.

Edge Disjoint Paths in Undirected Graphs

Problem

Given an undirected graph G, find the maximum number of edge disjoint paths in G

Reduction:

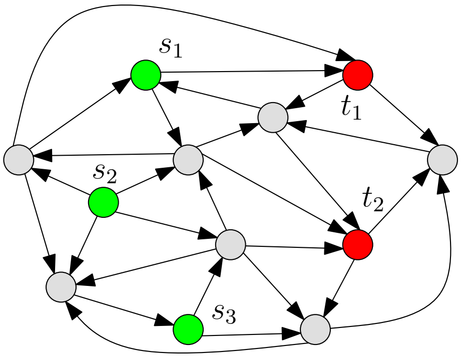
- create directed graph H by adding directed edges (u, v) and (v, u) for each edge uv in G.
- compute maximum s-t flow in H

Problem: Both edges (u, v) and (v, u) may have non-zero flow!

Not a Problem! Can assume maximum flow in H is acyclic and hence cannot have non-zero flow on both (u, v) and (v, u). Reduction works. See book for more details.

Multiple Sources and Sinks

- Directed graph G with edge capacities $c(e)$
- source nodes $s_1, s_2, \ldots, s_k$
- sink nodes $t_1, t_2, \ldots, t_\ell$
- sources and sinks are *disjoint*



Multiple Sources and Sinks

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- sink nodes $t_1, t_2, \ldots, t_\ell$
- sources and sinks are *disjoint*

Maximum Flow: send as much flow as possible from the sources to the sinks. *Sinks don't care which source they get flow from.*

Minimum Cut: find a minimum capacity set of edge E' such that removing E' disconnects every source from every sink.

Multiple Sources and Sinks: Formal Definition

- Directed graph G with edge capacities $c(e)$
- source nodes $s_1, s_2, \ldots, s_k$
- sink nodes $t_1, t_2, \ldots, t_\ell$
- sources and sinks are *disjoint*

A function $f : E \to \mathbb{R}^{\geq 0}$ is a flow if:

- for each $e \in E$, $f(e) \leq c(e)$ and
- for each v which is not a source or a sink $f^{\text{in}}(v) = f^{\text{out}}(v)$.

Goal: max $\sum_{i=1}^{k}(f^{\text{out}}(s_i) - f^{\text{in}}(s_i))$, that is, flow out of sources

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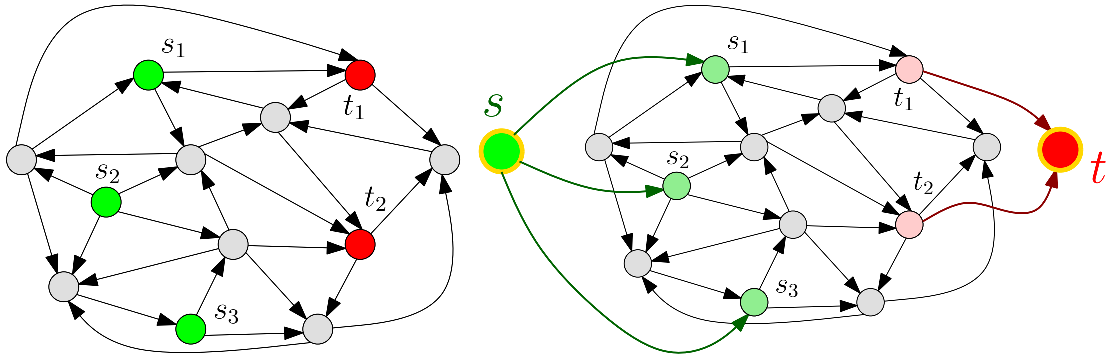
Reduction to Single-Source Single-Sink

- Add a *source* node s and a *sink* node t.
- Add edges $(s, s_1), (s, s_2), \ldots, (s, s_k)$.
- Add edges $(t_1, t), (t_2, t), \ldots, (t_\ell, t)$.
- Set the capacity of the new edges to be ∞.

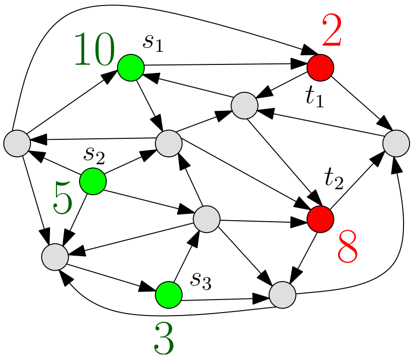


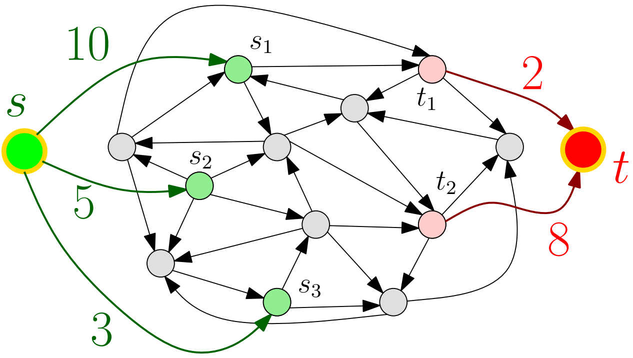
Supplies and Demands

A further generalization:

- source s_i has a supply of $S_i \geq 0$
- since t_j has a demand of $D_j \geq 0$ units

Question: is there a flow from source to sinks such that supplies are not exceeded and demands are met? Formally we have the additional constraints that $f^{\text{out}}(s_i) - f^{\text{in}}(s_i) \leq S_i$ for each source s_i and $f^{\text{in}}(t_j) - f^{\text{out}}(t_j) \leq D_j$ for each sink t_j.



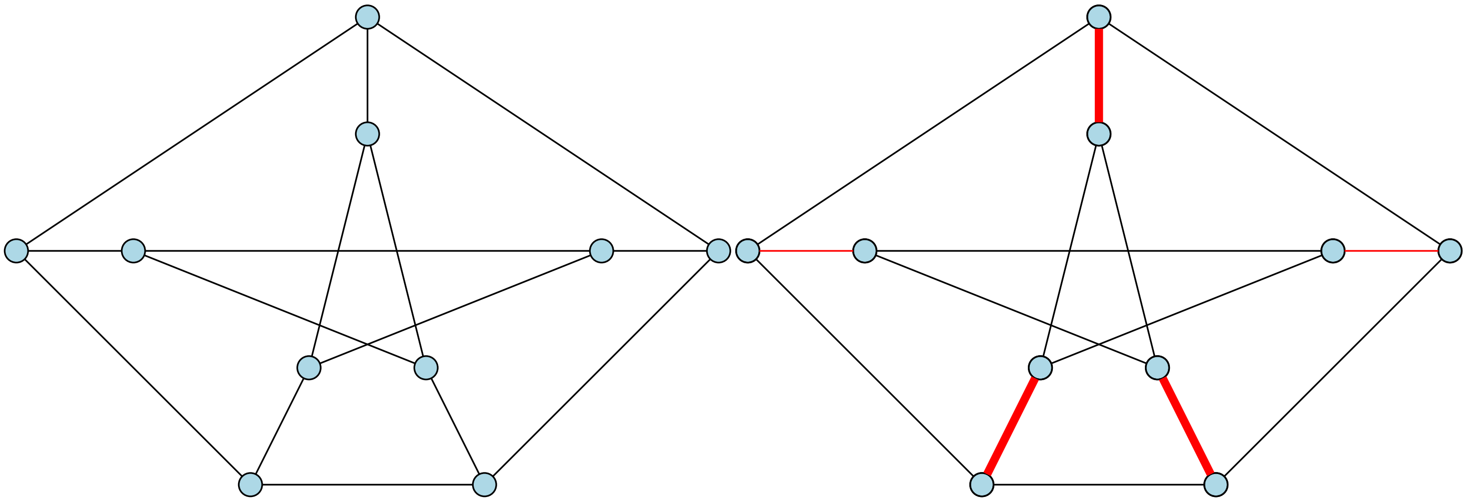


Matching

Input Given a (undirected) graph $G = (V, E)$

Goal Find a matching of maximum cardinality

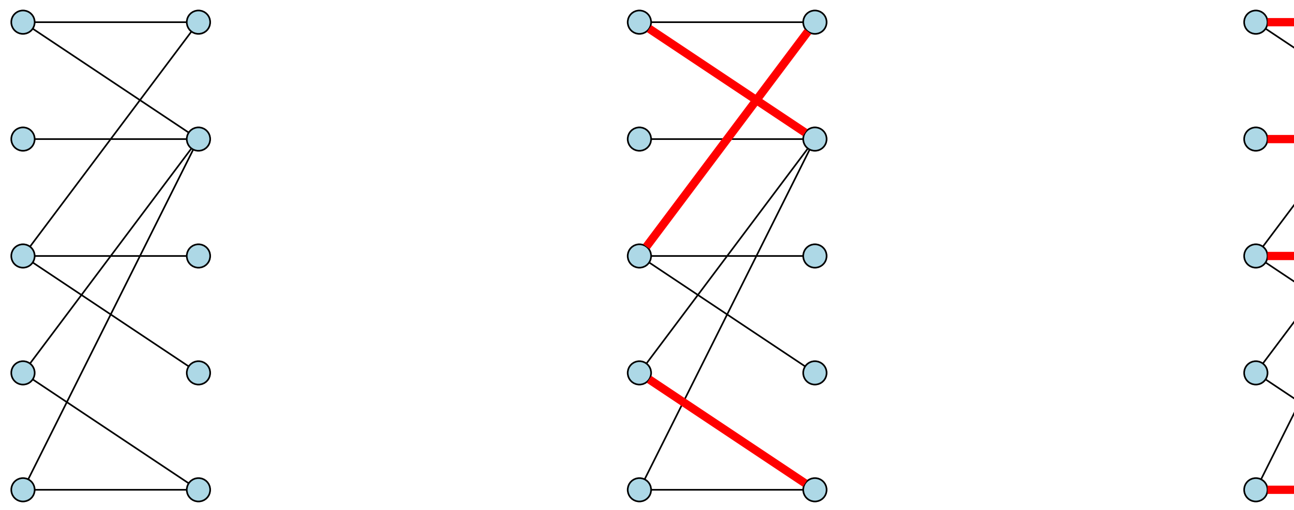
- A matching is $M \subseteq E$ such that at most one edge in M is incident on any vertex



Bipartite Matching

Input Given a bipartite graph $G = (L \cup R, E)$

Goal Find a matching of maximum cardinality

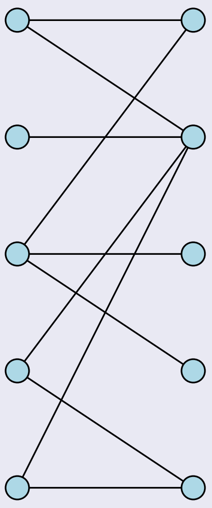


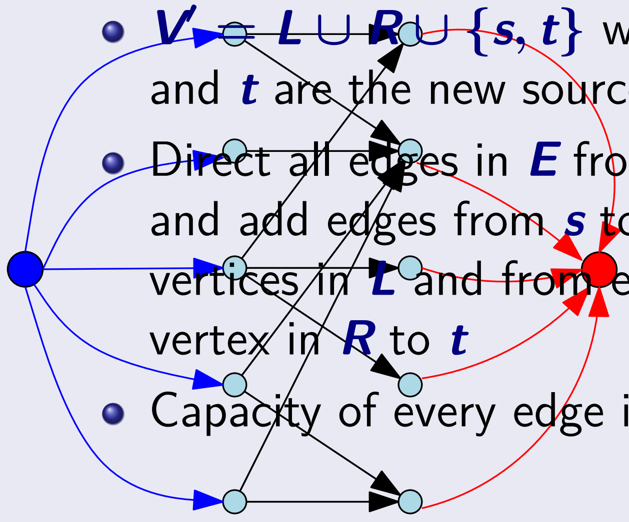
Maximum matching has 4 edges

Reduction to Max-Flow

Max-Flow Construction

Given graph $G = (L \cup R, E)$ create flow-network $G' = (V', E')$ as follows:





- $V' = L \cup R \cup \{s, t\}$ where s and t are the new source and sink
- Direct all edges in E from L to R, and add edges from s to all vertices in L and from each vertex in R to t
- Capacity of every edge is 1

Correctness: Matching to Flow

Proposition

If G has a matching of size k then G' has a flow of value k.

Proof.

Let M be matching of size k. Let $M = \{(u_1, v_1), \ldots, (u_k, v_k)\}$. Consider following flow f in G':

- $f(s, u_i) = 1$ and $f(v_i, t) = 1$ for $1 \leq i \leq k$
- $f(u_i, v_i) = 1$ for $1 \leq i \leq k$
- for all other edges flow is zero.

Verify that f is a flow of value k (because M is a matching). ☐

Correctness: Flow to Matching

Proposition

If G' has a flow of value k then G has a matching of size k.

Proof.

Consider flow f of value k.

- Can assume f is integral. Thus each edge has flow 1 or 0
- Consider the set M of edges from L to R that have flow 1
  - M has k edges because value of flow is equal to the number of non-zero flow edges crossing cut $(L \cup \{s\}, R \cup \{t\})$
  - Each vertex has at most one edge in M incident upon it. Why?

☐

Correctness of Reduction

Theorem

The maximum flow value in G' $=$ *maximum cardinality of matching in* G

Consequence

Thus, to find maximum cardinality matching in G, we construct G' and find the maximum flow in G'. Note that the matching itself (not just the value) can be found efficiently from the flow.

Running Time

For graph G with n vertices and m edges G' has $O(n+m)$ edges, and $O(n)$ vertices.

- Generic Ford-Fulkerson: Running time is $O(mC) = O(nm)$ since $C = n$
- Capacity scaling: Running time is $O(m^2 \log C) = O(m^2 \log n)$

Better known running time: $O(m\sqrt{n})$

Perfect Matchings

Definition

A matching M is said to be perfect if every vertex has one edge in M incident upon it.

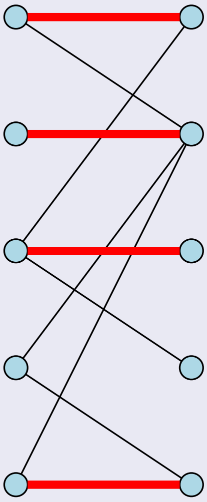


Figure: This graph does not have a perfect matching

Characterizing Perfect Matchings

Problem

When does a bipartite graph have a perfect matching?

- Clearly $|L| = |R|$
- Are there any necessary and sufficient conditions?

A Necessary Condition

Lemma

If $G = (L \cup R, E)$ has a perfect matching then for any $X \subseteq L$, $|N(X)| \geq |X|$, where $N(X)$ is the set of neighbors of vertices in X

Proof.

Since G has a perfect matching, every vertex of X is matched to a different neighbor, and so $|N(X)| \geq |X|$ □

Hall's Theorem

Theorem (Frobenius-Hall)

Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. G has a perfect matching if and only if for every $X \subseteq L$, $|N(X)| \geq |X|$

One direction is the necessary condition.

For the other direction we will show the following:

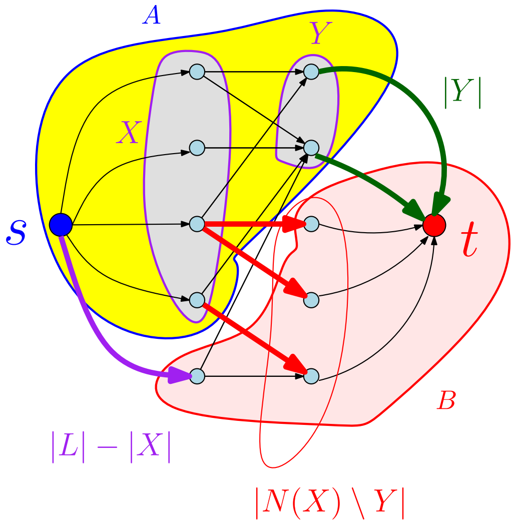
- create flow network G' from G
- if $|N(X)| \geq |X|$ for all X, show that minimum s-t cut in G' is of capacity $n = |L| = |R|$
- implies that G has a perfect matching

Proof of Sufficiency

Assume $|N(X)| \geq |X|$ for each $X \in L$. Then show that min s-t cut in G' is of capacity at least n.

Let (A, B) be an *arbitrary* s-t cut in G'

- let $X = A \cap L$ and $Y = A \cap R$
- cut capacity is at least $(|L| - |X|) + |Y| + |N(X) \setminus Y|$



Because there are...

- $|L| - |X|$ edges from s to $L \cap B$.
- $|Y|$ edges from Y to t.
- there are at least $|N(X) \setminus Y|$ edges from X to vertices on the right side that are not in Y.

Proof of Sufficiency

Continued...

- By the above, cut capacity is at least
$$\alpha = (|L| - |X|) + |Y| + |N(X) \setminus Y|.$$
- $|N(X) \setminus Y| \geq |N(X)| - |Y|$.
(This holds for any two sets.)
- By assumption $|N(X)| \geq |X|$ and hence
$$|N(X) \setminus Y| \geq |N(X)| - |Y| \geq |X| - |Y|.$$
- Cut capacity is therefore at least
$$\begin{aligned}\alpha &= (|L| - |X|) + |Y| + |N(X) \setminus Y| \\ &\geq |L| - |X| + |Y| + |X| - |Y| \geq |L| = n.\end{aligned}$$
- Any s-t cut capacity is at least $n \implies$ max flow at least n units $\implies$ perfect matching. **QED**

Application: assigning jobs to people

- n jobs or tasks
- m people
- for each job a set of people who can do that job
- for each person j a limit on number of jobs k_j
- Goal: find an assignment of jobs to people so that all jobs are assigned and no person is overloaded

Reduce to max-flow similar to matching.

Arises in many settings. Using *minimum-cost flows* can also handle the case when assigning a job i to person j costs c_{ij} and goal is assign all jobs but minimize cost of assignment.

Reduction to Maximum Flow

- Create directed graph $G = (V, E)$ as follows
  - $V = \{s, t\} \cup L \cup R$: L set of n jobs, R set of m people
  - add edges (s, i) for each job $i \in L$, capacity 1
  - add edges (j, t) for each person $j \in R$, capacity k_j
  - if job i can be done by person j add an edge (i, j), capacity 1
- Compute max s-t flow. There is an assignment if and only if flow value is n.

Matchings in General Graphs

Matchings in general graphs more complicated.

There is a polynomial time algorithm to compute a maximum matching in a general graph. Best known running time is $O(m\sqrt{n})$.