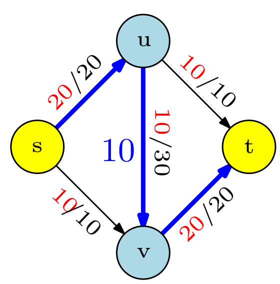
Chapter 17

Network Flow Algorithms

CS 473: Fundamental Algorithms, Fall 2011
October 27, 2011

17.1 Algorithm(s) for Maximum Flow

17.1.0.1 Greedy Approach



1. Begin with $f(e) = 0$ for each edge

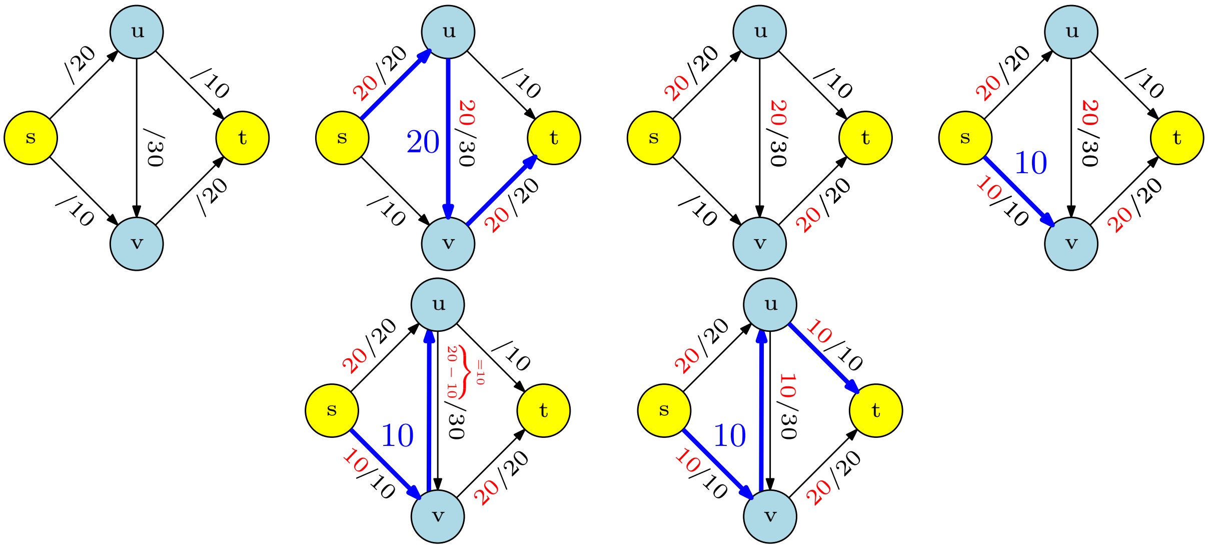
2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$

3. *Augment* flow along this path

4. Repeat augmentation for as long as possible.

17.1.1 Greedy Approach: Issues

17.1.1.1 Issues = What is this nonsense?



1. Begin with $f(e) = 0$ for each edge
2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$
3. Augment flow along this path
4. Repeat augmentation for as long as possible.

Greedy can get stuck in sub-optimal flow!
Need to "push-back" flow along edge (u, v)

17.2 Ford-Fulkerson Algorithm

17.2.1 Residual Graph

17.2.1.1 The "leftover" graph

Definition 17.2.1 *For a network $G = (V, E)$ and flow f, the residual graph $G_f = (V', E')$ of G with respect to f is*

[¡+-¿]

(A) $V' = V$

(B) Forward Edges: For each edge $e \in E$ with $f(e) < c(e)$, we add $e \in E'$ with capacity $c(e) - f(e)$

(C) Backward Edges: For each edge $e = (u, v) \in E$ with $f(e) > 0$, we add $(v, u) \in E'$ with capacity $f(e)$

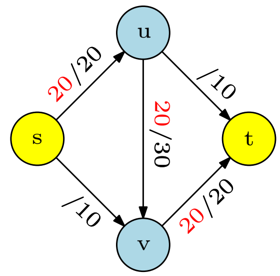


Figure 17.1: Flow on edges is indicated in red

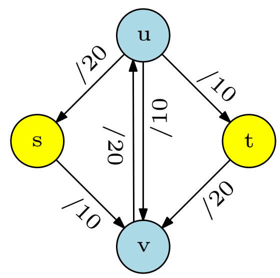


Figure 17.2: Residual Graph

17.2.1.2 Residual Graph Example
17.2.1.3 Residual Graph Property

Observation: Residual graph captures the "residual" problem exactly.

Lemma 17.2.2 *Let f be a flow in G and G_f be the residual graph. If f' is a flow in G_f then $f + f'$ is a flow in G of value $v(f) + v(f')$.*

Lemma 17.2.3 *Let f and f' be two flows in G with $v(f') \geq v(f)$. Then there is a flow f'' of value $v(f') - v(f)$ in G_f.*

Definition of + and - for flows is intuitive and the above lemmas are easy in some sense but a bit messy to formally prove.

17.2.1.4 Residual Graph Property: Implication

Recursive algorithm for finding a maximum flow:

MaxFlow(G, s, t):
    If the flow from s to t is 0
        return 0
    Find any flow f with $v(f) > 0$ in G
    Recursively compute a maximum flow f' in G_f
    Output the flow $f + f'$

Iterative algorithm for finding a maximum flow:

```
MaxFlow(G,s,t):
    Start with flow f that is 0 on all edges
    While there is a flow f' in G_f with v(f') > 0 do
        f = f + f'
        Update G_f
    endWhile
    Output f
```

17.2.1.5 Ford-Fulkerson Algorithm

```
algFordFulkerson
    for every edge e, f(e) = 0
    G_f is residual graph of G with respect to f
    while G_f has a simple s-t path do
        let P be simple s-t path in G_f
        f = augment(f, P)
        Construct new residual graph G_f
```

```
augment(f,P)
    let b be bottleneck capacity,
        i.e., min capacity of edges in P (in G_f)
    for each edge (u,v) in P do
        if e = (u,v) is a forward edge then
            f(e) = f(e) + b
        else (* (u,v) is a backward edge *)
            let e = (v,u) (* (v,u) is in G *)
            f(e) = f(e) - b
    return f
```

17.3 Correctness and Analysis

17.3.1 Termination

17.3.1.1 Properties about Augmentation: Flow

Lemma 17.3.1 *If f is a flow and P is a simple s-t path in G_f, then $f' = \texttt{augment}(f, P)$ is also a flow.*

Proof: Verify that f' is a flow. Let b be augmentation amount.

(A) *Capacity constraint:* If $(u, v) \in P$ is a forward edge then $f'(e) = f(e) + b$ and $b \leq c(e) - f(e)$. If $(u, v) \in P$ is a backward edge, then letting $e = (v, u)$, $f'(e) = f(e) - b$ and $b \leq f(e)$. Both cases $0 \leq f'(e) \leq c(e)$.

(B) *Conservation constraint:* Let v be an internal node. Let e_1, e_2 be edges of P incident to v. Four cases based on whether e_1, e_2 are forward or backward edges. Check cases (see fig next slide).

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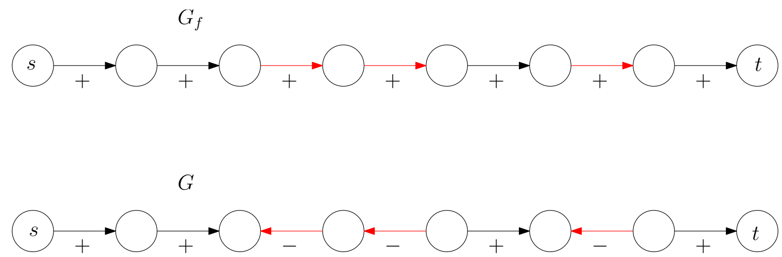


Figure 17.3: Augmenting path P in G_f and corresponding change of flow in G. Red edges are backward edges.

17.3.1.2 Properties about Augmentation: Conservation Constraint
17.3.1.3 Properties about Augmentation: Integer Flow

Lemma 17.3.2 *At every stage of the Ford-Fulkerson algorithm, the flow values $f(e)$ and the residual capacities in G_f are integers*

Proof: Initial flow and residual capacities are integers. Suppose lemma holds for j iterations. Then in $(j+1)$st iteration, minimum capacity edge b is an integer, and so flow after augmentation is an integer. ■

17.3.1.4 Progress in Ford-Fulkerson

Proposition 17.3.3 *Let f be a flow and f' be flow after one augmentation. Then $v(f) < v(f')$.*

Proof: Let P be an augmenting path, i.e., P is a simple s-t path in residual graph
(A) First edge e in P must leave s
(B) Original network G has no incoming edges to s; hence e is a forward edge
(C) P is simple and so never returns to s
(D) Thus, value of flow increases by the flow on edge e

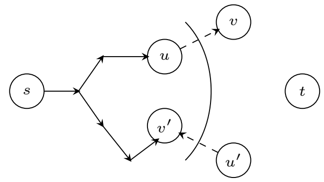
■

17.3.1.5 Termination Proof

Theorem 17.3.4 *Let C be the minimum cut value; in particular $C \leq \sum_{e \text{ out of } s} c(e)$. Ford-Fulkerson algorithm terminates after finding at most C augmenting paths.*

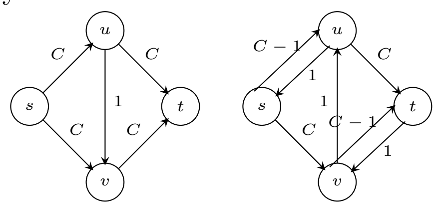
Proof: The value of the flow increases by at least 1 after each augmentation. Maximum value of flow is at most C. ■

Running time
(A) Number of iterations $\leq C$
(B) Number of edges in $G_f \leq 2m$
(C) Time to find augmenting path is $O(n+m)$
(D) Running time is $O(C(n+m))$ (or $O(mC)$).



17.3.1.6 Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the running time be as $\Omega(mC)$ or is our analysis weak?



Ford-Fulkerson can take $\Omega(C)$ iterations.

17.3.2 Correctness

17.3.2.1 Correctness of Ford-Fulkerson Augmenting Path Algorithm

Question: When the algorithm terminates, is the flow computed the maximum s-t flow?

Proof idea: show a cut of value equal to the flow. Also shows that maximum flow is equal to minimum cut!

17.3.2.2 Recalling Cuts

Definition 17.3.5 *Given a flow network an* **s-t cut** *is a set of edges $E' \subset E$ such that removing E'* disconnects *s from t: in other words there is no directed $s \to t$ path in $E - E'$.* Capacity *of cut E' is $\sum_{e \in E'} c(e)$.*

Let $A \subset V$ such that

(A) $s \in A$, $t \notin A$

(B) $B = V - A$ and hence $t \in B$

Define $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$

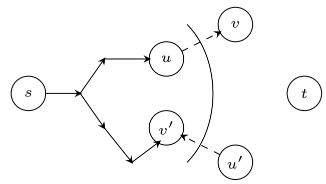
Claim 17.3.6 *(A, B) is an s-t cut.*

Recall: Every *minimal* s-t cut E' is a cut of the form (A, B).

17.3.2.3 Ford-Fulkerson Correctness

Lemma 17.3.7 *If there is no s-t path in G_f then there is some cut (A, B) such that $v(f) = c(A, B)$*

Proof: Let A be all vertices reachable from s in G_f; $B = V \setminus A$



(A) $s \in A$ and $t \in B$. So (A, B) is an s-t cut in G

(B) If $e = (u, v) \in G$ with $u \in A$ and $v \in B$, then $f(e) = c(e)$ (*saturated* edge) because otherwise v is reachable from s in G_f

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17.3.2.4 Lemma Proof Continued

Proof:

(A) If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s in G_f

(B) Thus,

$$
\begin{aligned}
v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\
&= f^{\text{out}}(A) - 0 \\
&= c(A, B) - 0 \\
&= c(A, B)
\end{aligned}
$$

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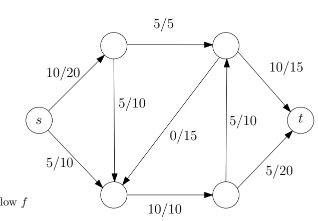
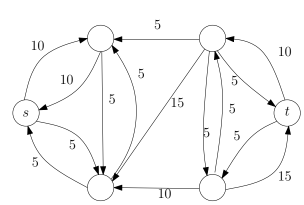
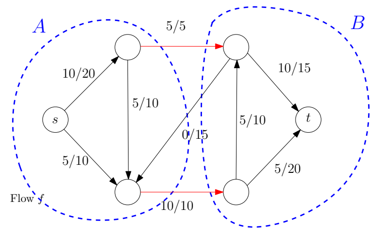
17.3.2.5 Example

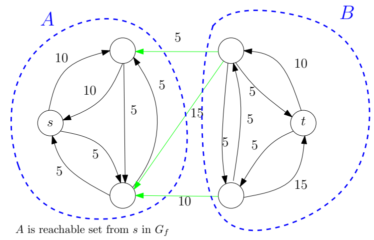


Flow f



Residual graph G_f: no s-t path





17.3.2.6 Ford-Fulkerson Correctness

Theorem 17.3.8 *The flow returned by the algorithm is the maximum flow.*

Proof:
(A) For any flow f and s-t cut (A, B), $v(f) \le c(A, B)$
(B) For flow f^* returned by algorithm, $v(f^*) = c(A^*, B^*)$ for some s-t cut (A^*, B^*)
(C) Hence, f^* is maximum

■

17.3.2.7 Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem 17.3.9 *For any network G, the value of a maximum s-t flow is equal to the capacity of the minimum s-t cut.*

Proof: Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. ■

17.3.2.8 Max-Flow Min-Cut Theorem and Integrality of Flows

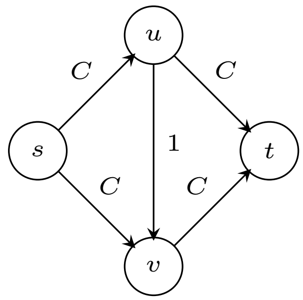
Theorem 17.3.10 *For any network G with integer capacities, there is a maximum s-t flow that is integer valued.*

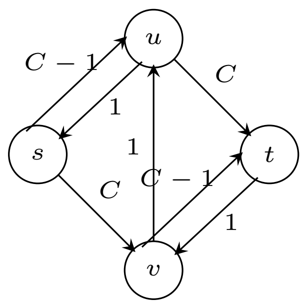
Proof: Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. ■

17.4 Polynomial Time Algorithms

17.4.0.9 Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the upper bound be achieved?





17.4.0.10 Polynomial Time Algorithms

Question: Is there a polynomial time algorithm for maxflow?

Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way? Yes! Two variants.

(A) Choose the augmenting path with largest bottleneck capacity.

(B) Choose the shortest augmenting path.

17.4.1 Capacity Scaling Algorithm

17.4.1.1 Augmenting Paths with Large Bottleneck Capacity

(A) Pick augmenting paths with largest bottleneck capacity in each iteration of Ford-Fulkerson

(B) How do we find path with largest bottleneck capacity?
   (A) Assume we know Δ the bottleneck capacity
   (B) Remove all edges with residual capacity $\leq \Delta$
   (C) Check if there is a path from s to t
   (D) Do binary search to find largest Δ
   (E) Running time: $O(m \log C)$

(C) Can we bound the number of augmentations? Can show that in $O(m \log C)$ augmentations the algorithm reaches a max flow. This leads to an $O(m^2 \log^2 C)$ time algorithm.

17.4.1.2 Augmenting Paths with Large Bottleneck Capacity

How do we find path with largest bottleneck capacity?

(A) Max bottleneck capacity is one of the edge capacities. Why?
(B) Can do binary search on the edge capacities. First, sort the edges by their capacities and then do binary search on that array as before.
(C) Algorithm's running time is $O(m \log m)$.
(D) Different algorithm that also leads to $O(m \log m)$ time algorithm by adapting Prim's algorithm.

17.4.1.3 Removing Dependence on C

(A) *[Edmonds-Karp, Dinitz]* Picking augmenting paths with fewest number of edges yields a $O(m^2 n)$ algorithm, i.e., independent of C. Such an algorithm is called a *strongly polynomial* time algorithm since the running time does not depend on the numbers (assuming RAM model). (Many implementation of Ford-Fulkerson would actually use shortest augmenting path if they use BFS to find an s-t path).
(B) Further improvements can yield algorithms running in $O(mn \log n)$, or $O(n^3)$.

17.4.1.4 Finding a Minimum Cut

Question: How do we find an actual minimum s-t cut?
Proof gives the algorithm!
(A) Compute an s-t maximum flow f in G
(B) Obtain the residual graph G_f
(C) Find the nodes A reachable from s in G_f
(D) Output the cut $(A, B) = \{(u, v) \mid u \in A, v \in B\}$. *Note:* The cut is found in G while A is found in G_f

Running time is essentially the same as finding a maximum flow.

Note: Given G and a flow f there is a linear time algorithm to check if f is a maximum flow and if it is, outputs a minimum cut. How?