Network Flow Algorithms

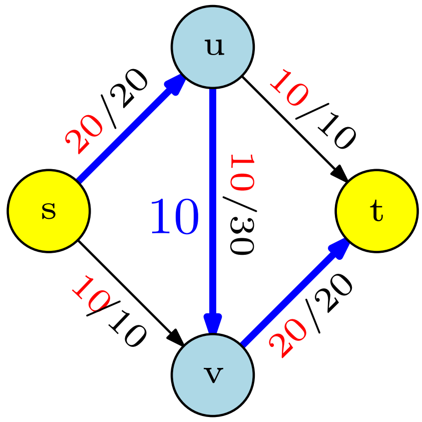
Lecture 17

October 27, 2011

Part I

Algorithm(s) for Maximum Flow

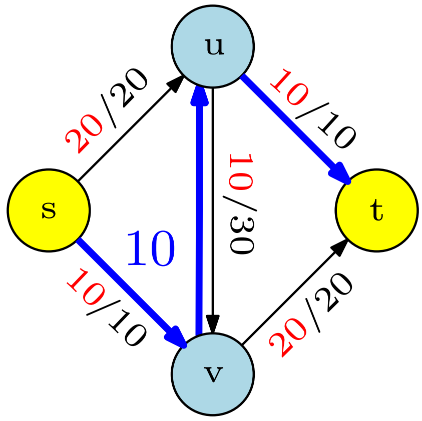
Greedy Approach



1. Begin with $f(e) = 0$ for each edge
2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$
3. *Augment* flow along this path
4. Repeat augmentation for as long as possible.

Greedy Approach: Issues

Issues = What is this nonsense?



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Greedy can get stuck in sub-optimal flow!
Need to "push-back" flow along edge (u, v)

Residual Graph

The "leftover" graph

Definition

For a network $G = (V, E)$ and flow f, the residual graph $G_f = (V', E')$ of G with respect to f is

- $V' = V$
- Forward Edges: For each edge $e \in E$ with $f(e) < c(e)$, we add $e \in E'$ with capacity $c(e) - f(e)$
- Backward Edges: For each edge $e = (u, v) \in E$ with $f(e) > 0$, we add $(v, u) \in E'$ with capacity $f(e)$

Residual Graph Example

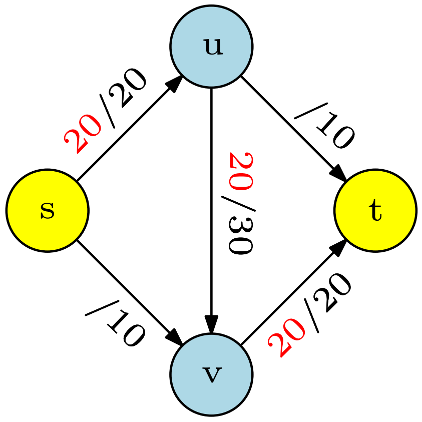


Figure: Flow on edges is indicated in red

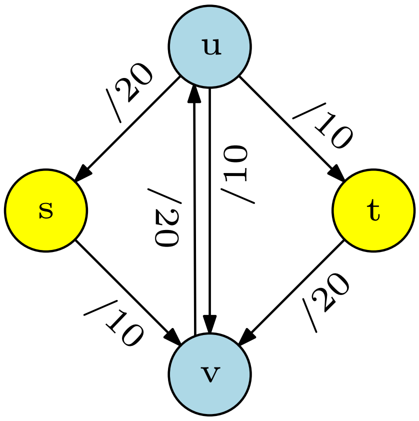


Figure: Residual Graph

Residual Graph Property

Observation: Residual graph captures the "residual" problem exactly.

Lemma

Let f be a flow in G and G_f be the residual graph. If f' is a flow in G_f then $f + f'$ is a flow in G of value $v(f) + v(f')$.

Lemma

Let f and f' be two flows in G with $v(f') \geq v(f)$. Then there is a flow f'' of value $v(f') - v(f)$ in G_f.

Definition of $+$ and - for flows is intuitive and the above lemmas are easy in some sense but a bit messy to formally prove.

Residual Graph Property: Implication

Recursive algorithm for finding a maximum flow:

```
MaxFlow($G, s, t$):
    If the flow from $s$ to $t$ is $0$
        return $0$
    Find any flow $f$ with $v(f) > 0$ in $G$
    Recursively compute a maximum flow $f'$ in $G_f$
    Output the flow $f + f'$
```

Iterative algorithm for finding a maximum flow:

```
MaxFlow($G, s, t$):
    Start with flow $f$ that is $0$ on all edges
    While there is a flow $f'$ in $G_f$ with $v(f') > 0$ do
        $f$ = $f + f'$
        Update $G_f$
    endWhile
    Output $f$
```

Ford-Fulkerson Algorithm

```
algFordFulkerson
    for every edge $e$, $f(e) = 0$
    $G_f$ is residual graph of $G$ with respect to $f$
    while $G_f$ has a simple $s$-$t$ path do
        let $P$ be simple $s$-$t$ path in $G_f$
        $f = \mathrm{augment}(f, P)$
        Construct new residual graph $G_f$
```

```
augment($f, P$)
    let $b$ be bottleneck capacity,
        i.e., min capacity of edges in $P$ (in $G_f$)
    for each edge $(u, v)$ in $P$ do
        if $e = (u, v)$ is a forward edge then
            $f(e) = f(e) + b$
        else (* $(u, v)$ is a backward edge *)
            let $e = (v, u)$ (* $(v, u)$ is in $G$ *)
            $f(e) = f(e) - b$
    return $f$
```

Properties about Augmentation: Flow

Lemma

If f is a flow and P is a simple s-t path in G_f, then $f' = \texttt{augment}(f, P)$ is also a flow.

Proof.

Verify that f' is a flow. Let b be augmentation amount.

- Capacity constraint: If $(u, v) \in P$ is a forward edge then $f'(e) = f(e) + b$ and $b \leq c(e) - f(e)$. If $(u, v) \in P$ is a backward edge, then letting $e = (v, u)$, $f'(e) = f(e) - b$ and $b \leq f(e)$. Both cases $0 \leq f'(e) \leq c(e)$.
- Conservation constraint: Let v be an internal node. Let e_1, e_2 be edges of P incident to v. Four cases based on whether e_1, e_2 are forward or backward edges. Check cases (see fig next slide). □

Properties about Augmentation: Conservation Constraint

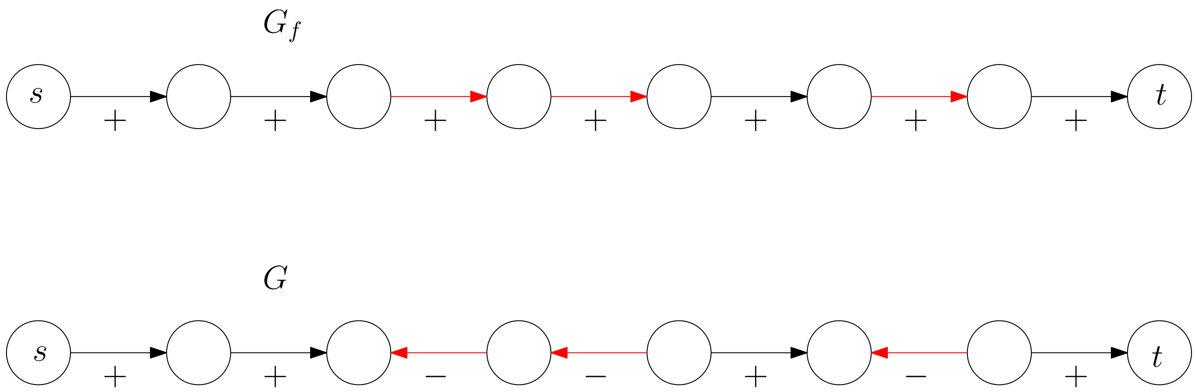


Figure: Augmenting path P in G_f and corresponding change of flow in G. Red edges are backward edges.

Properties about Augmentation: Integer Flow

Lemma

At every stage of the Ford-Fulkerson algorithm, the flow values $f(e)$ and the residual capacities in G_f are integers

Proof.

Initial flow and residual capacities are integers. Suppose lemma holds for j iterations. Then in $(j + 1)$st iteration, minimum capacity edge b is an integer, and so flow after augmentation is an integer. □

Progress in Ford-Fulkerson

Proposition

Let f be a flow and f' be flow after one augmentation. Then $v(f) < v(f')$.

Proof.

Let P be an augmenting path, i.e., P is a simple s-t path in residual graph

- First edge e in P must leave s
- Original network G has no incoming edges to s; hence e is a forward edge
- P is simple and so never returns to s
- Thus, value of flow increases by the flow on edge e □

Termination Proof

Theorem

Let C be the minimum cut value; in particular $C \leq \sum_{e \text{ out of } s} c(e)$. Ford-Fulkerson algorithm terminates after finding at most C augmenting paths.

Proof.

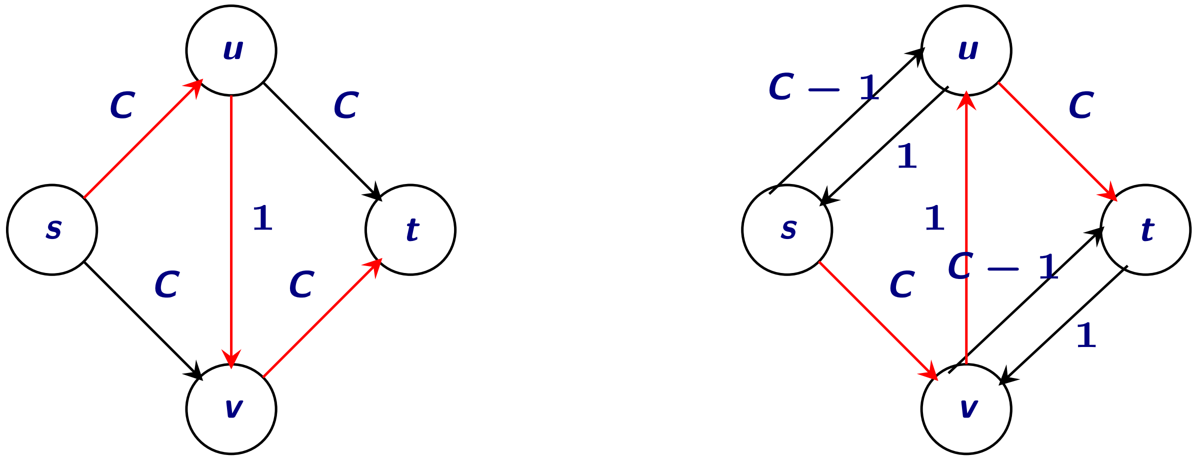
The value of the flow increases by at least 1 after each augmentation. Maximum value of flow is at most C. □

Running time

- Number of iterations $\leq C$
- Number of edges in $G_f \leq 2m$
- Time to find augmenting path is $O(n + m)$
- Running time is $O(C(n + m))$ (or $O(mC)$).

Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the running time be as $\Omega(mC)$ or is our analysis weak?



Ford-Fulkerson can take $\Omega(C)$ iterations.

Correctness of Ford-Fulkerson Augmenting Path Algorithm

Question: When the algorithm terminates, is the flow computed the maximum s-t flow?

Proof idea: show a cut of value equal to the flow. Also shows that maximum flow is equal to minimum cut!

Recalling Cuts

Definition

Given a flow network an **s-t cut** is a set of edges $E' \subset E$ such that removing E' *disconnects* s from t: in other words there is no directed $s \to t$ path in $E - E'$. *Capacity* of cut E' is $\sum_{e \in E'} c(e)$.

Let $A \subset V$ such that

- $s \in A$, $t \notin A$
- $B = V - A$ and hence $t \in B$

Define $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$

Claim

(A, B) *is an* s-t *cut.*

Recall: Every *minimal* s-t cut E' is a cut of the form (A, B).

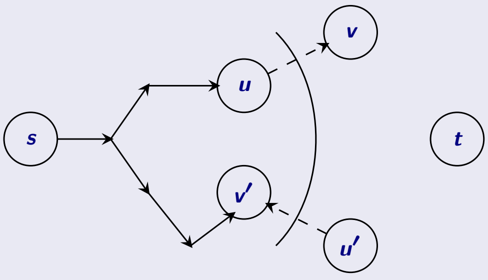
Ford-Fulkerson Correctness

Lemma

If there is no s-t *path in* G_f *then there is some cut* (A, B) *such that* $v(f) = c(A, B)$

Proof.

Let A be all vertices reachable from s in G_f; $B = V \setminus A$



- $s \in A$ and $t \in B$. So (A, B) is an s-t cut in G
- If $e = (u, v) \in G$ with $u \in A$ and $v \in B$, then $f(e) = c(e)$ (saturated edge) because otherwise v is reachable from s in G_f

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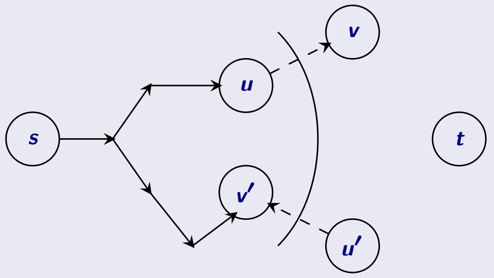
Lemma Proof Continued

Proof.

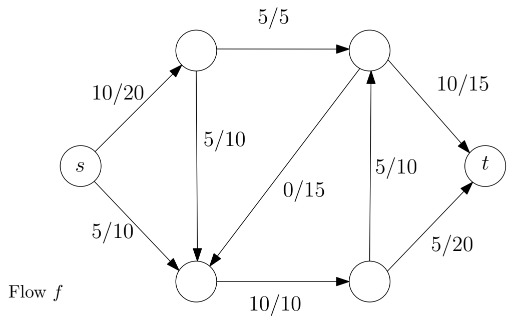


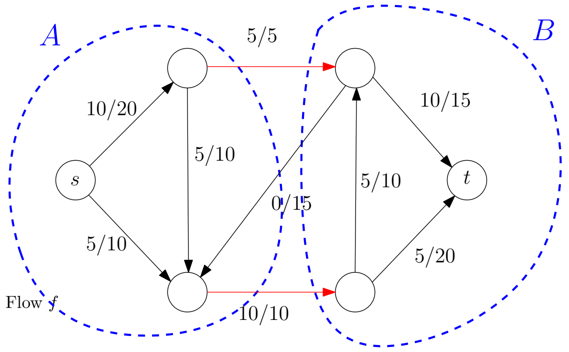
- If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s in G_f
- Thus,

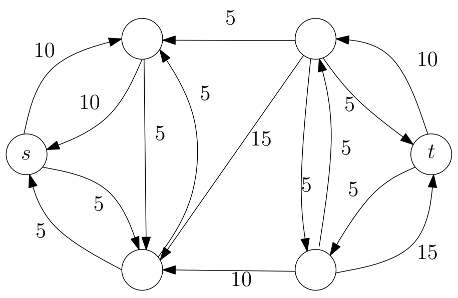
$$\begin{aligned} v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\ &= f^{\text{out}}(A) - 0 \\ &= c(A, B) - 0 \\ &= c(A, B) \end{aligned}$$

□

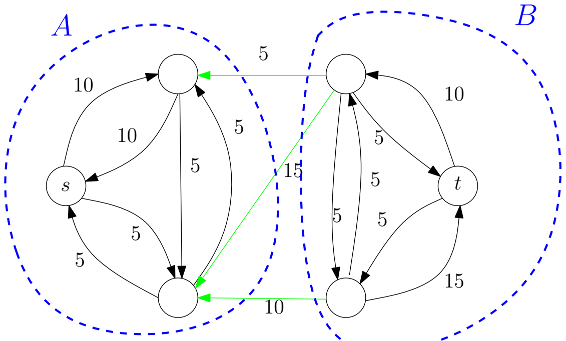
Example







Residual graph G_f: no s-t path



A is reachable set from s in G_f

Ford-Fulkerson Correctness

Theorem

The flow returned by the algorithm is the maximum flow.

Proof.

- For any flow f and s-t cut (A, B), $v(f) \leq c(A, B)$
- For flow f^* returned by algorithm, $v(f^*) = c(A^*, B^*)$ for some s-t cut (A^*, B^*)
- Hence, f^* is maximum

□

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network G, the value of a maximum s-t flow is equal to the capacity of the minimum s-t cut.

Proof.

Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. □

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

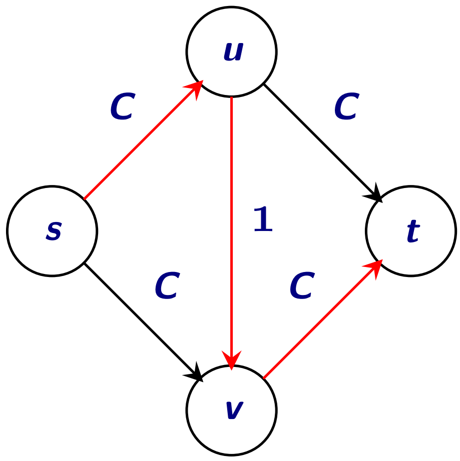
For any network G *with integer capacities, there is a maximum* s-t *flow that is integer valued.*

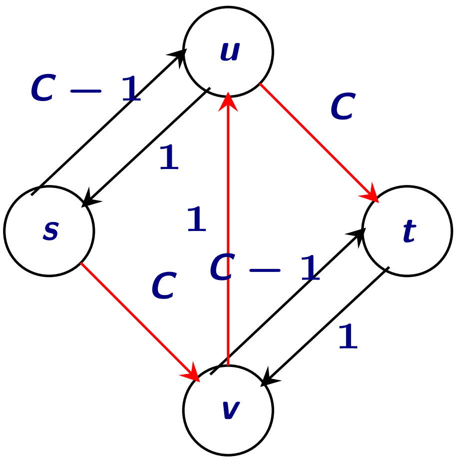
Proof.

Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. □

Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the upper bound be achieved?





Polynomial Time Algorithms

Question: Is there a polynomial time algorithm for maxflow?

Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way? Yes! Two variants.

- Choose the augmenting path with largest bottleneck capacity.
- Choose the shortest augmenting path.

Augmenting Paths with Large Bottleneck Capacity

- Pick augmenting paths with largest bottleneck capacity in each iteration of Ford-Fulkerson
- How do we find path with largest bottleneck capacity?
  - Assume we know Δ the bottleneck capacity
  - Remove all edges with residual capacity $\leq \Delta$
  - Check if there is a path from s to t
  - Do binary search to find largest Δ
  - Running time: $O(m \log C)$
- Can we bound the number of augmentations? Can show that in $O(m \log C)$ augmentations the algorithm reaches a max flow. This leads to an $O(m^2 \log^2 C)$ time algorithm.

Augmenting Paths with Large Bottleneck Capacity

How do we find path with largest bottleneck capacity?

- Max bottleneck capacity is one of the edge capacities. Why?
- Can do binary search on the edge capacities. First, sort the edges by their capacities and then do binary search on that array as before.
- Algorithm's running time is $O(m \log m)$.
- Different algorithm that also leads to $O(m \log m)$ time algorithm by adapting Prim's algorithm.

Removing Dependence on C

- [Edmonds-Karp, Dinitz] Picking augmenting paths with fewest number of edges yields a $O(m^2 n)$ algorithm, i.e., independent of C. Such an algorithm is called a strongly polynomial time algorithm since the running time does not depend on the numbers (assuming RAM model). (Many implementation of Ford-Fulkerson would actually use shortest augmenting path if they use BFS to find an s-t path).
- Further improvements can yield algorithms running in $O(mn \log n)$, or $O(n^3)$.

Finding a Minimum Cut

Question: How do we find an actual minimum s-t cut?
Proof gives the algorithm!

- Compute an s-t maximum flow f in G
- Obtain the residual graph G_f
- Find the nodes A reachable from s in G_f
- Output the cut $(A, B) = \{(u, v) \mid u \in A, v \in B\}$. Note: The cut is found in G while A is found in G_f

Running time is essentially the same as finding a maximum flow.

Note: Given G and a flow f there is a linear time algorithm to check if f is a maximum flow and if it is, outputs a minimum cut. How?