

Network Flows

Lecture 16

October 25, 2011

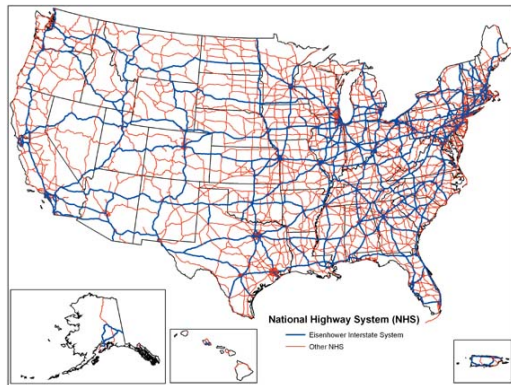
Everything flows

Panta rei – everything flows (literally).
Heraclitus (535–475 BC)

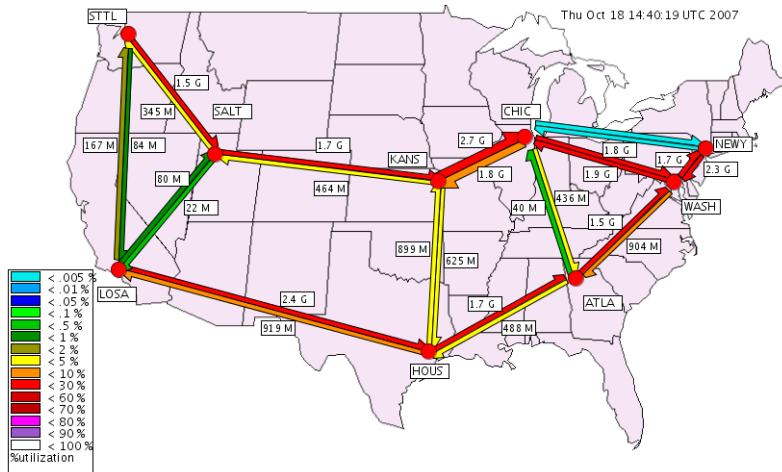
Part I

Network Flows: Introduction and Setup

Transportation/Road Network



Internet Backbone Network



Common Features of Flow Networks

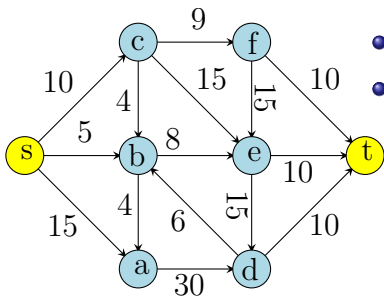
- *Network* represented by a (directed) *graph* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
- Each edge \mathbf{e} has a *capacity* $\mathbf{c}(\mathbf{e}) \geq 0$ that limits amount of *traffic* on \mathbf{e}
- *Source(s)* of traffic/data
- *Sink(s)* of traffic/data
- Traffic *flows* from sources to sinks
- Traffic is *switched/interchanged* at nodes

Flow: abstract term to indicate stuff (traffic/data/etc) that *flows* from sources to sinks.

Single Source Single Sink Flows

Simple setting:

- single source s and single sink t
- every other node v is an *internal* node
- flow originates at s and terminates at t



- Each edge e has a capacity $c(e) \geq 0$
- Some times it is convenient to assume that source $s \in V$ has no incoming edges and sink $t \in V$ has no outgoing edges

Assumptions: All capacities are integer, and every vertex has at least one edge incident to it.

Definition of Flow

Two ways to define flows:

- edge based
- path based

They are essentially equivalent but have different uses.

Edge based definition is more compact.

Edge Based Definition of Flow

Definition

A **flow** in a network $G = (V, E)$, is a function $f: E \rightarrow \mathbb{R}^{\geq 0}$ such that

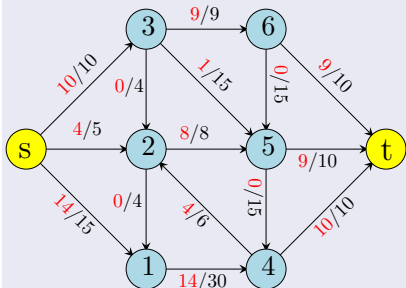


Figure: Flow with value

- **Capacity Constraint:** For each edge e , $f(e) \leq c(e)$
- **Conservation Constraint:** For each vertex $v \neq s, t$

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

- **Value of flow:** (total flow out of source) – (total flow in to source)

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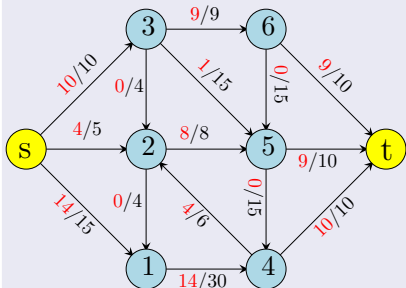


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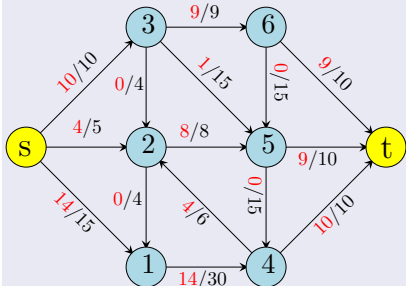


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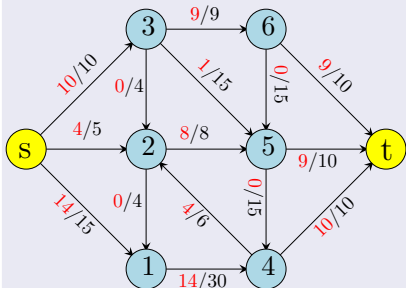


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Conservation of flow law is also known as *Kirchhoff's law*.

More Definitions and Notation

Notation

- The inflow into a vertex v is $f^{\text{in}}(v) = \sum_{e \text{ into } v} f(e)$ and the outflow is $f^{\text{out}}(v) = \sum_{e \text{ out of } v} f(e)$
- For a set of vertices A , $f^{\text{in}}(A) = \sum_{e \text{ into } A} f(e)$. Outflow $f^{\text{out}}(A)$ is defined analogously

Definition

For a network $G = (V, E)$ with source s , the **value** of flow f is defined as $v(f) = f^{\text{out}}(s) - f^{\text{in}}(s)$

A Path Based Definition of Flow

Intuition: flow goes from source s to sink t along a path.

\mathcal{P} : set of all paths from s to t . $|\mathcal{P}|$ can be *exponential* in n .

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- **Conservation Constraint:** No need! Automatic.

Value of flow: $\sum_{p \in \mathcal{P}} f(p)$

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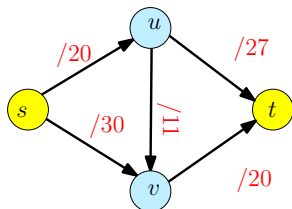
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Example



$$\mathcal{P} = \{p_1, p_2, p_3\}$$

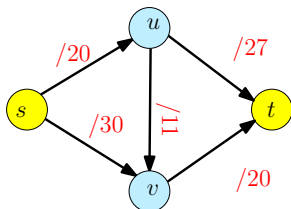
$$p_1 : s \rightarrow u \rightarrow t$$

$$p_2 : s \rightarrow u \rightarrow v \rightarrow t$$

$$p_3 : s \rightarrow v \rightarrow t$$

$$f(p_1) = 10, f(p_2) = 4, f(p_3) = 6$$

Example



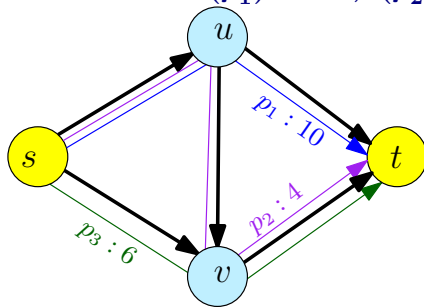
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Path based flow implies Edge based flow

Lemma

Given a path based flow $f: \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ there is an edge based flow $f': E \rightarrow \mathbb{R}^{\geq 0}$ of the same value.

Proof.

For each edge e define $f'(e) = \sum_{p: e \in p} f(p)$.

Exercise: verify capacity and conservation constraints for f' .

Exercise: verify that value of f and f' are equal □

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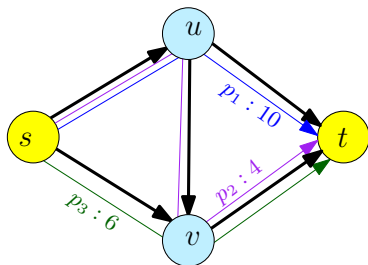
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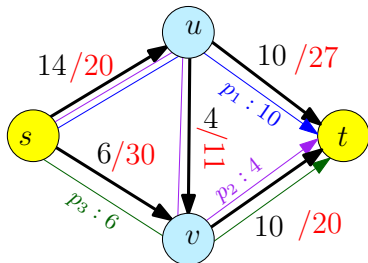
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$$f(p_1) = 10, f(p_2) = 4, f(p_3) = 6$$



$$f((s, u)) = 14$$

$$f((u, v)) = 4$$

$$f((s, v)) = 6$$

$$f((u, t)) = 10$$

$$f((v, t)) = 10$$

Flow Decomposition

Edge based flow to Path based Flow

Lemma

Given an edge based flow $f' : E \rightarrow \mathbb{R}^{\geq 0}$, there is a path based flow $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ of same value. Moreover, f assigns non-negative flow to at most m paths where $|E| = m$ and $|V| = n$. Given f' , the path based flow can be computed in $O(mn)$ time.

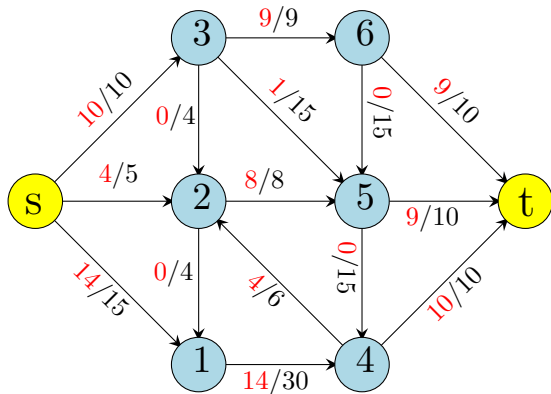
Flow Decomposition

Edge based flow to Path based Flow

Proof Idea.

- remove all edges with $f(e) = 0$
- find a path p from s to t
- assign $f(p)$ to be $\min_{e \in p} f(e)$
- reduce $f(e)$ for all $e \in p$ by $f(p)$
- repeat until no path from s to t
- in each iteration at least one edge has flow reduced to zero; hence at most m iterations. Can be implemented in $O(m(m+n))$ time. $O(mn)$ time requires care. □

Example



Edge vs Path based Definitions of Flow

Edge based flows:

- *compact* representation, only m values to be specified
- need to check flow conservation explicitly at each internal node

Path flows:

- in some applications, paths more natural,
- not compact,
- no need to check flow conservation constraints.

Equivalence shows that we can go back and forth easily.

The Maximum-Flow Problem

Problem

Input A network G with capacity c and source s and sink t

Goal Find flow of *maximum* value

Question: Given a flow network, what is an *upper bound* on the maximum flow between source and sink?

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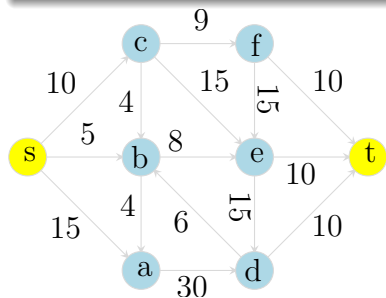
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Cuts

Definition (s-t cut)

Given a flow network an **s-t cut** is a set of edges $E' \subset E$ such that removing E' disconnects s from t : in other words there is no directed $s \rightarrow t$ path in $E - E'$.

The **capacity** of a cut E' is $c(E') = \sum_{e \in E'} c(e)$.



Caution:

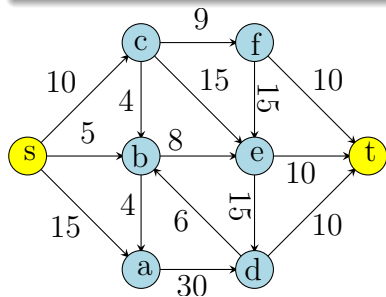
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- There might be many $s-t$ cuts.

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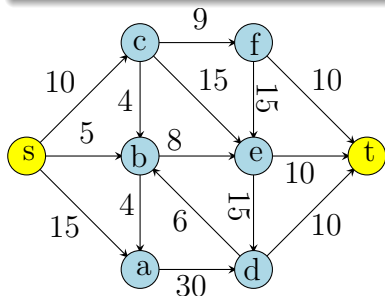
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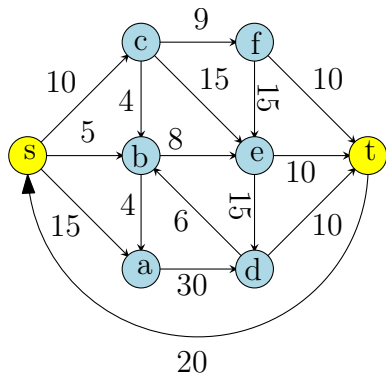
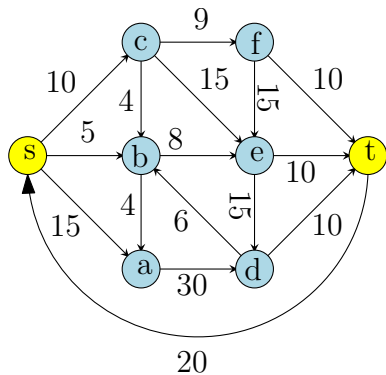


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s — t cuts

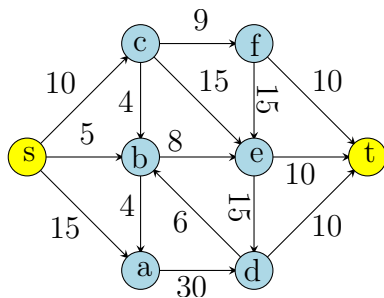
A death by a thousand cuts



Minimal Cut

Definition

Given a flow network an s - t , E' is a **minimal cut** if for all $e \in E'$, $E' - \{e\}$ is not a cut.

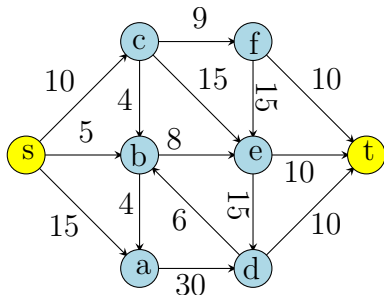


Observation: given a cut E' , can check efficiently whether E' is a minimal cut or not. How?

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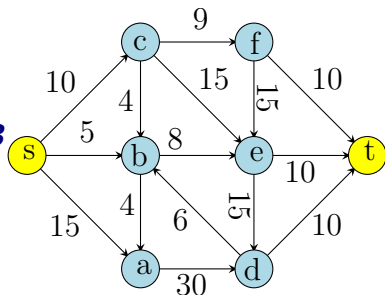
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Cuts as Vertex Partitions

Let $A \subset V$ such that

- $s \in A, t \notin A$
- $B = V - A$ and hence $t \in B$

Define $\text{cut}(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$
The set of edges leaving A .



Claim

(A, B) is an s - t cut.

Proof.

Let P be any $s \rightarrow t$ path in G . Since t is not in A , P has to leave A via some edge (u, v) in (A, B) . \square

Cuts as Vertex Partitions

Lemma

Suppose E' is an s - t cut. Then there is a cut (A, B) such that $(A, B) \subseteq E'$.

Proof.

E' is an s - t cut implies no path from s to t in $(V, E - E')$.

- Let A be set of all nodes reachable by s in $(V, E - E')$.
- Since E' is a cut, $t \notin A$.
- $(A, B) \subseteq E'$. Why? If some edge $(u, v) \in (A, B)$ is not in E' then v will be reachable by s and should be in A , hence a contradiction. □

Corollary

Every minimal s - t cut E' is a cut of the form (A, B) .

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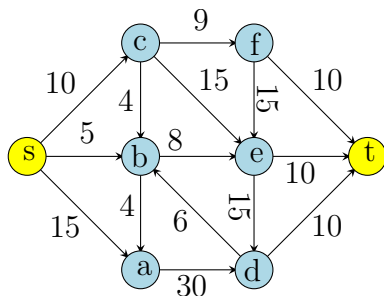
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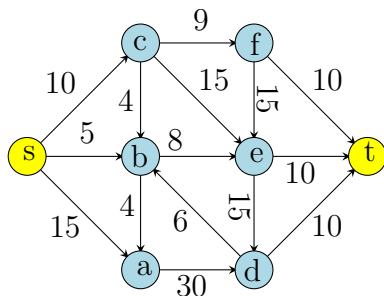


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Input A flow network G

Goal Find the capacity of a *minimum* s - t cut

Flows and Cuts

Lemma

For any s - t cut E' , **maximum s - t flow** \leq capacity of E' .

Proof.

Formal proof easier with path based definition of flow.

Suppose $f: \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ is a max-flow.

Every path $p \in \mathcal{P}$ contains an edge $e \in E'$. Why?

Assign each path $p \in \mathcal{P}$ to exactly one edge $e \in E'$.

Let \mathcal{P}_e be paths assigned to $e \in E'$. Then

$$v(f) = \sum_{p \in \mathcal{P}} f(p) = \sum_{e \in E'} \sum_{p \in \mathcal{P}_e} f(p) \leq \sum_{e \in E'} c(e)$$



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Corollary

Maximum s - t flow \leq minimum s - t cut.

Max-Flow Min-Cut Theorem

Theorem

*In any flow network the maximum **s-t** flow is equal to the minimum **s-t** cut.*

Can compute minimum-cut from maximum flow and vice-versa!
Proof coming shortly.

Many applications:

- optimization
- graph theory
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Exercise: Given G, s, t as above, show that one can remove all edges into s and all edges out of t without affecting the flow value between s and t .

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