More Dynamic Programming

Lecture 9

February 17, 2011

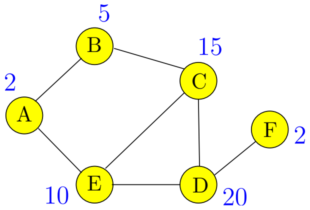
Part I

Maximum Weighted Independent Set in Trees

Maximum Weight Independent Set Problem

Input Graph $G = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal Find maximum weight independent set in G



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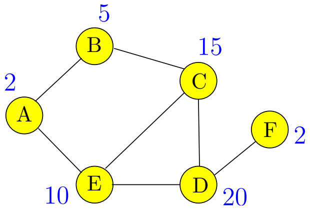

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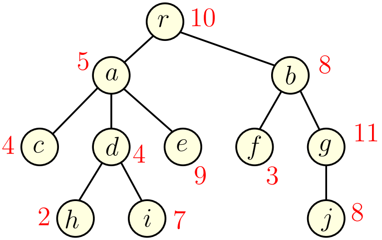
Maximum Weight Independent Set in a Tree

Input Tree $T = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal Find maximum weight independent set in T



Maximum weight independent set in above tree: ??

Towards a Recursive Solution

For an arbitrary graph G:

- Number vertices as $v_1, v_2, \ldots, v_n$
- Find recursively optimum solutions without v_n (recurse on $G - v_n$) and with v_n (recurse on $G - v_n - N(v_n)$ & include v_n).
- Saw that if graph G is arbitrary there was no good ordering that resulted in a small number of subproblems.

What about a tree? Natural candidate for v_n is root r of T?

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Natural candidate for v_n is root r of T? Let $\mathcal{O}$ be an optimum solution to the whole problem.

Case $r \notin \mathcal{O}$: Then $\mathcal{O}$ contains an optimum solution for each subtree of T hanging at a child of r.

Case $r \in \mathcal{O}$: None of the children of r can be in $\mathcal{O}$. $\mathcal{O} - \{r\}$ contains an optimum solution for each subtree of T hanging at a grandchild of r.

Subproblems? Subtrees of T hanging at nodes in T.

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A Recursive Solution

$T(u)$: subtree of T hanging at node u
$OPT(u)$: max weighted independent set value in $T(u)$

$$OPT(u) = \max \begin{cases} \sum_{v \text{ child of } u} OPT(v), \\ w(u) + \sum_{v \text{ grandchild of } u} OPT(v) \end{cases}$$

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Iterative Algorithm

- Compute $\boldsymbol{OPT}(\boldsymbol{u})$ bottom up. To evaluate $\boldsymbol{OPT}(\boldsymbol{u})$ need to have computed values of all children and grandchildren of $\boldsymbol{u}$
- What is an ordering of nodes of a tree $\boldsymbol{T}$ to achieve above?
  Post-order traversal of a tree.

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- Compute $OPT(u)$ bottom up. To evaluate $OPT(u)$ need to have computed values of all children and grandchildren of u
- What is an ordering of nodes of a tree T to achieve above? Post-order traversal of a tree.

Iterative Algorithm

MIS-Tree(T):
    Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of T
    for $i = 1$ to n **do**
        $M[v_i] = \max\left(\sum_{v_j \text{ child of } v_i} M[v_j],\ \ w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j]\right)$
    return $M[v_n]$ (* Note: v_n is the root of T *)

Space: $O(n)$ to store the value at each node of T
Running time:

- Naive bound: $O(n^2)$ since each $M[v_i]$ evaluation may take $O(n)$ time and there are n evaluations.
- Better bound: $O(n)$. A value $M[v_j]$ is accessed only by its parent and grand parent.

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Let v_1, v_2, ..., v_n be a post-order traversal of nodes of T
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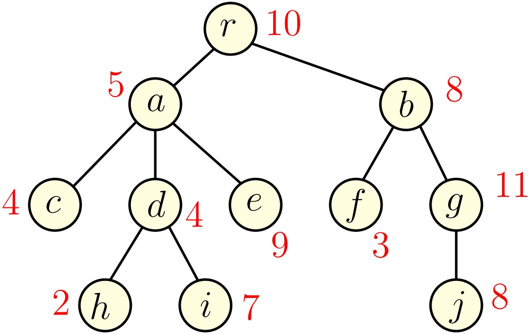

Part II

DAGs and Dynamic Programming

Recursion and DAGs

Observation

Let A be a recursive algorithm for problem Π. For each instance I of Π there is an associated DAG $G(I)$.

- Create directed graph $G(I)$ as follows...
- For each sub-problem in the execution of A on I create a node.
- If sub-problem v *depends* on or *recursively calls* sub-problem u add *directed* edge (u, v) to graph.
- $G(I)$ is a DAG. Why? If $G(I)$ has a cycle then A will not terminate on I.

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Iterative Algorithm for...

Dynamic Programming and DAGs

Observation

*An iterative algorithm **B** obtained from a recursive algorithm **A** for a problem Π does the following:*

> *For each instance **I** of Π, it computes a topological sort of $G(I)$ and evaluates sub-problems according to the topological ordering.*

- Sometimes the DAG $G(I)$ can be obtained directly without thinking about the recursive algorithm **A**
- In some cases (not all) the computation of an optimal solution reduces to a shortest/longest path in DAG $G(I)$
- Topological sort based shortest/longest path computation is dynamic programming!

Let O_i be value of an optimal schedule for the first i jobs.

```
Schedule(n):
    if n = 0 then return 0
    if n = 1 then return w(v_1)
    O_p(n) <-Schedule(p(n))
    O_n-1 <-Schedule(n - 1)
    if (O_p(n) + w(v_n) < O_n-1) then
        O_n = O_n-1
    else
        O_n = O_p(n) + w(v_n)
    return O_n
```

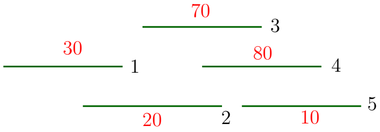
Weighted Interval Scheduling via...

Longest Path in a DAG

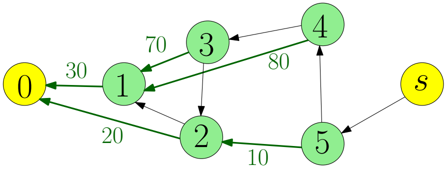
Given intervals, create a DAG as follows:

- Create one node for each interval, plus a dummy sink node 0 for interval 0, plus a dummy source node s.
- For each interval i add edge $(i, p(i))$ of the length/weight of v_i.
- Add an edge from s to n of length 0.
- For each interval i add edge $(i, i-1)$ of length 0.

Example



$p(5) = 2,\ p(4) = 1,\ p(3) = 1,\ p(2) = 0,\ p(1) = 0$



Relating Optimum Solution

Given interval problem instance I let $G(I)$ denote the DAG constructed as described.

Claim

Optimum solution to weighted interval scheduling instance I is given by longest *path from s to 0 in $G(I)$.*

Assuming claim is true,

- If I has n intervals, DAG $G(I)$ has $n+2$ nodes and $O(n)$ edges. Creating $G(I)$ takes $O(n \log n)$ time: to find $p(i)$ for each i. How?
- Longest path can be computed in $O(n)$ time — recall $O(m+n)$ algorithm for shortest/longest paths in DAGs.

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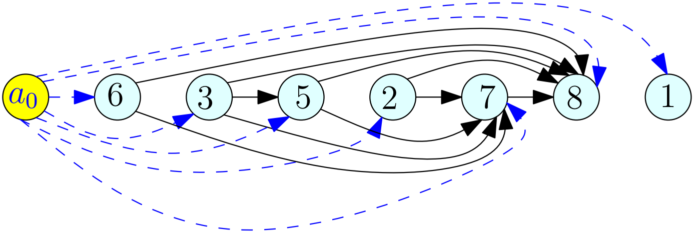
DAG for Longest Increasing Sequence

Given sequence $a_1, a_2, \ldots, a_n$ create DAG as follows:

- add sentinel a_0 to sequence where a_0 is less than smallest element in sequence
- for each i there is a node v_i
- if $i < j$ and $a_i < a_j$ add an edge (v_i, v_j)
- find longest path from v_0



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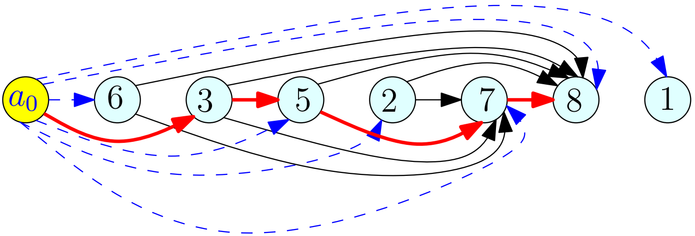

Part III

Edit Distance and Sequence Alignment

Spell Checking Problem

Given a string "exponen" that is not in the dictionary, how should a spell checker suggest a *nearby* string?

What does nearness mean?

Question: Given two strings $x_1 x_2 \ldots x_n$ and $y_1 y_2 \ldots y_m$ what is a *distance* between them?

Edit Distance: minimum number of "edits" to transform x into y.

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Edit Distance

Definition

Edit distance between two words X and Y is the number of letter insertions, letter deletions and letter substitutions required to obtain Y from X.

Example

The edit distance between FOOD and MONEY is at most 4:

$$\underline{\mathbf{F}}\mathbf{OOD} \rightarrow \mathbf{MO}\underline{\mathbf{O}}\mathbf{D} \rightarrow \mathbf{MON}\underline{\mathbf{O}}\mathbf{D} \rightarrow \mathbf{MONE}\underline{\mathbf{D}} \rightarrow \mathbf{MONEY}$$

Edit Distance: Alternate View

Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

F	**O**	**O**		**D**
M	**O**	**N**	**E**	**Y**

Formally, an alignment is a set M of pairs (i, j) such that each index appears at most once, and there is no "crossing": $i < i'$ and i is matched to j implies i' is matched to $j' > j$. In the above example, this is $M = \{(1, 1), (2, 2), (3, 3), (4, 5)\}$. Cost of an alignment is the number of mismatched columns plus number of unmatched indices in both strings.

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Edit Distance Problem

Problem

Given two words, find the edit distance between them, i.e., an alignment of smallest cost.

Applications

- Spell-checkers and Dictionaries
- Unix `diff`
- DNA sequence alignment . . . but, we need a new metric

Similarity Metric

Definition

For two strings X and Y, the cost of alignment M is

- [Gap penalty] For each gap in the alignment, we incur a cost δ.
- [Mismatch cost] For each pair p and q that have been matched in M, we incur cost α_{pq}; typically $\alpha_{pp} = 0$.

Edit distance is special case when $\delta = \alpha_{pq} = 1$.

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An Example

Example

o		c	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

Cost $= \delta + \alpha_{ae}$

Alternative:

o		c	u	r	r		a	n	c	e
o	c	c	u	r	r	e		n	c	e

Cost $= 3\delta$

Or a really stupid solution (delete string, insert other string):

o	c	u	r	r	a	n	c	e										
									o	c	c	u	r	r	e	n	c	e

Cost $= 19\delta$.

Sequence Alignment

Input Given two words X and Y, and gap penalty δ and mismatch costs α_{pq}

Goal Find alignment of minimum cost

Edit distance

Basic observation

Let $X = \alpha x$ and $Y = \beta y$

α, β: stings.

x and y single characters.

Think about optimal edit distance between X and Y as alignment, and consider last column of alignment of the two strings:

α	x
β	y

or

α	x
βy	

or

αx	
β	y

Observation

Prefixes must have optimal alignment!

Problem Structure

Observation

Let $X = x_1 x_2 \cdots x_m$ and $Y = y_1 y_2 \cdots y_n$. If (m, n) are not matched then either the mth position of X remains unmatched or the nth position of Y remains unmatched.

- Case x_m and y_n are matched.
  - Pay mismatch cost $\alpha_{x_m y_n}$ plus cost of aligning strings $x_1 \cdots x_{m-1}$ and $y_1 \cdots y_{n-1}$
- Case x_m is unmatched.
  - Pay gap penalty plus cost of aligning $x_1 \cdots x_{m-1}$ and $y_1 \cdots y_n$
- Case y_n is unmatched.
  - Pay gap penalty plus cost of aligning $x_1 \cdots x_m$ and $y_1 \cdots y_{n-1}$

Subproblems and Recurrence

Optimal Costs

Let $\mathrm{Opt}(i, j)$ be optimal cost of aligning $x_1 \cdots x_i$ and $y_1 \cdots y_j$. Then

$$\mathrm{Opt}(i,j) = \min \begin{cases} \alpha_{x_i y_j} + \mathrm{Opt}(i-1, j-1), \\ \delta + \mathrm{Opt}(i-1, j), \\ \delta + \mathrm{Opt}(i, j-1) \end{cases}$$

Base Cases: $\mathrm{Opt}(i, 0) = \delta \cdot i$ and $\mathrm{Opt}(0, j) = \delta \cdot j$

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Dynamic Programming Solution

for all i do $M[i,0] = i\delta$
for all j do $M[0,j] = j\delta$

for $i = 1$ to m do
    for $j = 1$ to n do

$$M[i,j] = \min \begin{cases} \alpha_{x_i y_j} + M[i-1, j-1], \\ \delta + M[i-1, j], \\ \delta + M[i, j-1] \end{cases}$$

Analysis

- Running time is $O(mn)$.
- Space used is $O(mn)$.

Dynamic Programming Solution

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Matrix and DAG of Computation

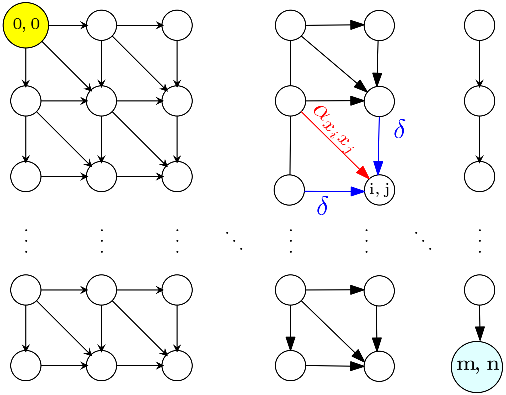

Figure: Iterative algorithm in previous slide computes values in row order. Optimal value is a shortest path from $(0, 0)$ to (m, n) in .

Sequence Alignment in Practice

- Typically the DNA sequences that are aligned are about 10^5 letters long!
- So about 10^{10} operations and 10^{10} bytes needed
- The killer is the 10GB storage
- Can we reduce space requirements?

Optimizing Space

- Recall

$$M(i,j) = \min \begin{cases} \alpha_{x_i y_j} + M(i-1, j-1), \\ \delta + M(i-1, j), \\ \delta + M(i, j-1) \end{cases}$$

- Entries in jth column only depend on $(j-1)$st column and earlier entries in jth column
- Only store the current column and the previous column reusing space; $N(i,0)$ stores $M(i,j-1)$ and $N(i,1)$ stores $M(i,j)$

Computing in column order to save space

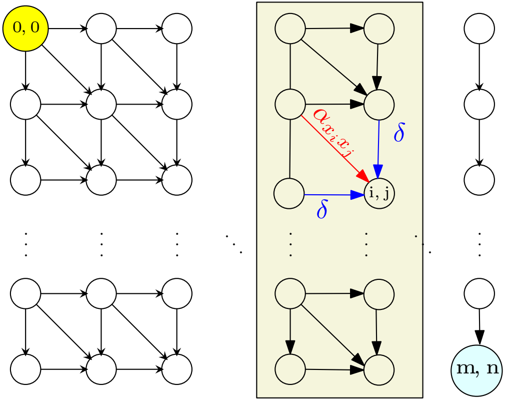

Figure: $M(i, j)$ only depends on previous column values. Keep only two columns and compute in column order.

Space Efficient Algorithm

```
for all i do N[i,0] = iδ
for j = 1 to n do
    N[0,1] = jδ (* corresponds to M(0,j) *)
    for i = 1 to m do
```

$$N[i,1] = \min \begin{cases} \alpha_{x_i y_j} + N[i-1,0] \\ \delta + N[i-1,1] \\ \delta + N[i,0] \end{cases}$$

```
    for i = 1 to m do
        Copy N[i,0] = N[i,1]
```

Analysis

Running time is $O(mn)$ and space used is $O(2m) = O(m)$

Analyzing Space Efficiency

- From the $\boldsymbol{m} \times \boldsymbol{n}$ matrix $\boldsymbol{M}$ we can construct the actual alignment (exercise)
- Matrix $\boldsymbol{N}$ computes cost of optimal alignment but no way to construct the actual alignment
- Space efficient computation of alignment? More complicated algorithm — see text book.

Takeaway Points

- Dynamic programming is based on finding a recursive way to solve the problem. Need a recursion that generates a small number of subproblems.
- Given a recursive algorithm there is a natural DAG associated with the subproblems that are generated for given instance; this is the dependency graph. An iterative algorithm simply evaluates the subproblems in some topological sort of this DAG.
- The space required to evaluate the answer can be reduced in some cases by a careful examination of that dependency DAG of the subproblems and keeping only a subset of the DAG at any time.

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