Dynamic Programming

Lecture 8

February 15, 2011

Part I

Longest Increasing Subsequence

Sequences

Definition

Sequence: an ordered list $a_1, a_2, \ldots, a_n$. ***Length*** of a sequence is number of elements in the list.

Definition

$a_{i_1}, \ldots, a_{i_k}$ is a ***subsequence*** of $a_1, \ldots, a_n$ if $1 \leq i_1 < \ldots < i_k \leq n$.

Definition

A sequence is ***increasing*** if $a_1 < a_2 < \ldots < a_n$. It is ***non-decreasing*** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly ***decreasing*** and non-increasing.

Sequences

Example...

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Subsequence: 5, 2, 1
- Increasing sequence: 3, 5, 9
- Increasing subsequence: 2, 7, 8

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an *increasing subsequence* $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

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Naïve Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

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Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

$\mathrm{LIS}(A[1..n])$:

- Case 1: does not contain a_n in which case $\mathrm{LIS}(A[1..n]) = \mathrm{LIS}(A[1..(n-1)])$
- Case 2: contains a_n in which case $\mathrm{LIS}(A[1..n])$ is not so clear.

Observation: if a_n is in the longest increasing subsequence then all the elements before it must be smaller.

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algLIS(A[1..n]):
    if (n = 0) then return 0
    m = algLIS(A[1..(n − 1)])
    B is subsequence of A[1..(n − 1)] with
            only elements less than a_n
    (* let h be size of B, h ≤ n-1 *)
    m = max(m, 1 + algLIS(B[1..h]))
    Output m
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Recursion for running time: $T(n) \leq 2T(n-1) + O(n)$.
Easy to see that $T(n)$ is $O(n2^n)$.

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Observation

For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than a_n. This suggests that a more general problem is **LIS_smaller**$(A[1..n], x)$ *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Recursive Approach: Take 2

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

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LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
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```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
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Question: Is there any advantage?

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Observation: The number of *different* subproblems generated by **LIS_smaller**$(A[1..n], x)$ is $O(n^2)$. Memoization the recursive algorithm leads to an $O(n^2)$ running time!

Question: What are the recursive subproblem generated by **LIS_smaller**$(A[1..n], x)$?

- For $0 \leq i < n$ **LIS_smaller**$(A[1..i], y)$ where y is either x or one of $A[i+1], \ldots, A[n]$.

Observation: previous recursion also generates only $O(n^2)$ subproblems. Slightly harder to see.

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Recursive Algorithm: Take 3

LISEnding$(A[1..n])$: length of longest increasing sub-sequence that *ends* in $A[n]$.

Question: can we obtain a recursive expression?

$$\textbf{LISEnding}(A[1..n]) = \max_{i:A[i]<A[n]} \left(1 + \textbf{LISEnding}(A[1..i])\right)$$

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LIS_ending(A[1..n]):
    if (n = 0) return 0
    m = 1
    for i = 1 to n − 1 do
        if (A[i] < A[n]) then
            m = max(m, 1 + LIS_ending(A[1..i]))

    return m
```

```
LIS(A[1..n]):
    return max_{i=1}^{n} LIS_ending(A[1 . . . i])
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Question:

How many distinct subproblems generated by LISEnding($A[1..n]$)? n.

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Iterative Algorithm via Memoization

Compute the values **LISEnding**$(A[1..i])$ iteratively in a bottom up fashion.

```
LIS_ending(A[1..n]):
    Array L[1..n]
        (* L[i] = value of LIS_ending(A[1..i]) *)
    for i = 1 to n do
        L[i] = 1
        for j = 1 to i - 1 do
            if (A[j] < A[i]) do
                L[i] = max(L[i], 1 + L[j])
    return L
```

```
LIS(A[1..n]):
    L = LIS_ending(A[1..n])
    return the maximum value in L
```

Iterative Algorithm via Memoization

Simplifying:

```
LIS(A[1..n]):
    Array L[1..n]
          (* L[i] stores the value LIS-Ending(A[1..i]) *)
    m = 0
    for i = 1 to n do
        L[i] = 1
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            if (A[j] < A[i]) do
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Correctness: Via induction following the recursion

Running time: $O(n^2)$, Space: $\Theta(n)$

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- $L[i]$ is value of longest increasing subsequence ending in $A[i]$
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Memoizing LIS_smaller

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LIS(A[1..n]):
    A[n+1] = ∞ (* add a sentinel at the end *)
    Array L[(n+1),(n+1)] (* two-dimensional array*)
        (* L[i,j] for j ≥ i stores the value LIS_smaller(A[1..i],A[j]) *)
    for j = 1 to n+1 do
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    return L[n,(n+1)]
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```
LIS(A[1..n]):
    A[n + 1] = ∞ (* add a sentinel at the end *)
    Array L[(n + 1), (n + 1)] (* two-dimensional array*)
        (* L[i, j] for j ≥ i stores the value LIS_smaller(A[1..i], A[j]) *)
    for j = 1 to n + 1 do
        L[0, j] = 0
    for i = 1 to n + 1 do
        for j = i to n + 1 do
            L[i, j] = L[i − 1, j]
            if (A[i] < A[j]) then
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    return L[n, (n + 1)]
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Longest increasing subsequence

Another way to get quadratic time algorithm

- $G = (\{s, 1, \ldots, n\}, \{\})$: directed graph.
  - $\forall i, j$: If $i < j$ and $A[i] < A[j]$ then
    add the edge $i \rightarrow j$ to G.
  - $\forall i$: Add $s \rightarrow i$.
- The graph G is a DAG. LIS corresponds to longest path in G starting at s.
- We know how to compute this in $O(|V(G)| + |E(G)|) = O(n^2)$.

Comment: One can compute LIS in $O(n \log n)$ time with a bit more work.

Dynamic Programming

- Find a “smart” recursion for the problem in which the number of distinct subproblems is small; polynomial in the original problem size.
- Eliminate recursion and find an iterative algorithm to compute the problems bottom up by storing the intermediate values in an appropriate data structure; need to find the right way or order the subproblem evaluation.
- Estimate the number of subproblems, the time to evaluate each subproblem and the space needed to store the value. Evaluate the total running time.
- Optimize the resulting algorithm further

Part II

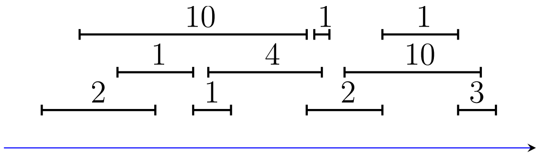
Weighted Interval Scheduling

Weighted Interval Scheduling

Input A set of jobs with start times, finish times and *weights* (or profits).

Goal Schedule jobs so that total weight of jobs is maximized.

- Two jobs with overlapping intervals cannot both be scheduled!



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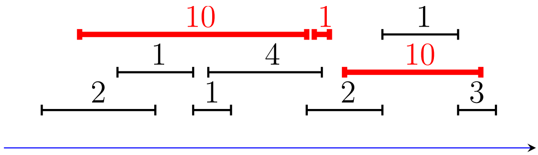
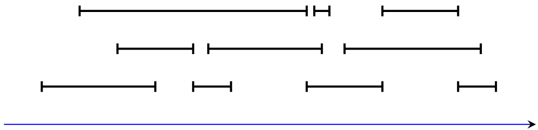


Input A set of jobs with start and finish times to be scheduled on a resource; special case where all jobs have weight 1.

Goal Schedule as many jobs as possible.

- Greedy strategy of considering jobs according to finish times produces optimal schedule (to be seen later).



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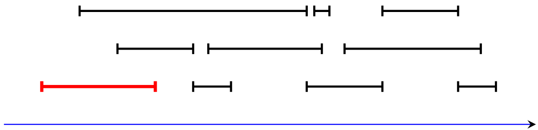

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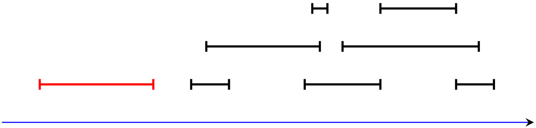

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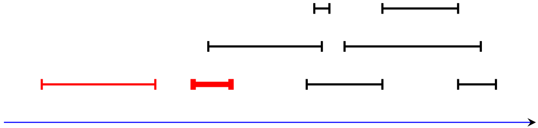

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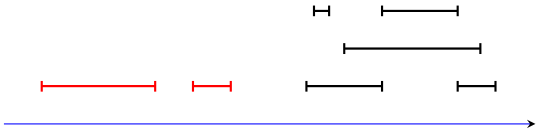

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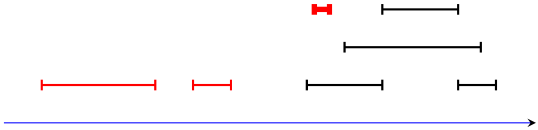

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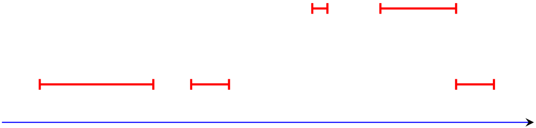

Greedy Strategies

- Earliest finish time first
- Largest weight/profit first
- Largest weight to length ratio first
- Shortest length first
- ...

None of the above strategies lead to an optimum solution.

Moral: Greedy strategies often don't work!

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Reduction to...

Max Weight Independent Set Problem

- Given weighted interval scheduling instance I create an instance of max weight independent set on a graph $G(I)$ as follows.
  - For each interval i create a vertex v_i with weight w_i.
  - Add an edge between v_i and v_j if i and j overlap.
- Claim: max weight independent set in $G(I)$ has weight equal to max weight set of intervals in I that do not overlap
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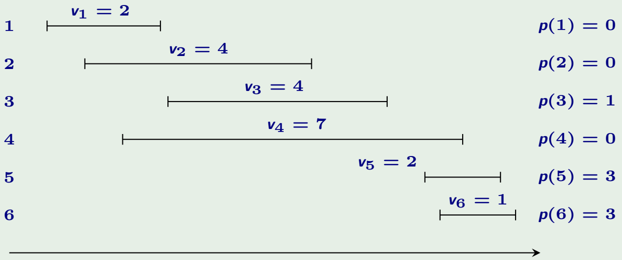
Conventions

Definition

- Let the requests be sorted according to finish time, i.e., $i < j$ implies $f_i \leq f_j$
- Define $p(j)$ to be the largest i (less than j) such that job i and job j are not in conflict

Example



Towards a Recursive Solution

Observation

Consider an optimal schedule $\mathcal{O}$

Case $n \in \mathcal{O}$: *None of the jobs between n and $p(n)$ can be scheduled. Moreover $\mathcal{O}$ must contain an optimal schedule for the first $p(n)$ jobs.*

Case $n \not\in \mathcal{O}$: *$\mathcal{O}$ is an optimal schedule for the first $n - 1$ jobs.*

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Case $n \notin \mathcal{O}$: $\mathcal{O}$ *is an optimal schedule for the first* $n - 1$ *jobs.*

A Recursive Algorithm

Let O_i be value of an optimal schedule for the first i jobs.

```
Schedule(n):
    if n = 0 then return 0
    if n = 1 then return w(v_1)
    O_p(n) <- Schedule(p(n))
    O_{n-1} <- Schedule(n - 1)
    if (O_p(n) + w(v_n) < O_{n-1}) then
        O_n = O_{n-1}
    else
        O_n = O_p(n) + w(v_n)
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```

Time Analysis

Running time is $T(n) = T(p(n)) + T(n-1) + O(1)$ which is ...

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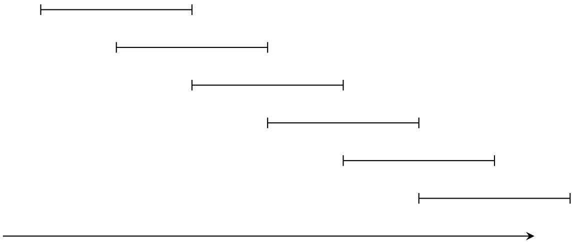

Figure: Bad instance for recursive algorithm

Running time on this instance is

$$T(n) = T(n-1) + T(n-2) + O(1) = \Theta(\phi^n)$$

where $\phi \approx 1.618$ is the golden ratio.

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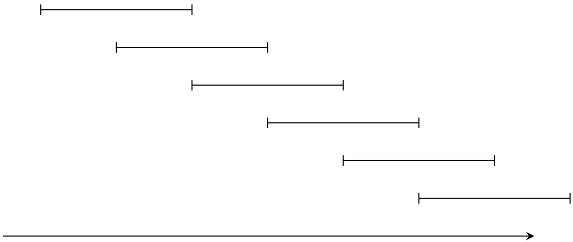

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Analysis of the Problem

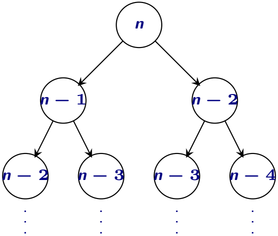

Figure: Label of node indicates size of sub-problem. Tree of sub-problems grows very quickly

Memo(r)ization

Observation

- *Number of different sub-problems in recursive algorithm is* $O(n)$*; they are* $O_1, O_2, \ldots, O_{n-1}$
- *Exponential time is due to recomputation of solutions to sub-problems*

Solution

Store optimal solution to different sub-problems, and perform recursive call only if not already computed.

Recursive Solution with Memoization

```
schdIMem(j)
    if j = 0 then return 0
    if M[j] is defined then (* sub-problem already solved *)
        return M[j]
    if M[j] is not defined then
        M[j] = max( w(v_j) + schdIMem(p(j)),   schdIMem(j - 1) )
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```

Time Analysis

- Each invocation, $O(1)$ time plus: either return a computed value, or generate 2 recursive calls and fill one $M[\cdot]$
- Initially no entry of $M[]$ is filled; at the end all entries of $M[]$ are filled.
- So total time is $O(n)$ (Assuming input is presorted...)

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Automatic Memoization

Fact

Many functional languages (like LISP) automatically do memoization for recursive function calls!

Back to Weighted Interval Scheduling

Iterative Solution

$$
\boxed{\begin{array}{l}
M[0] = 0 \\
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M: table of subproblems

- Implicitly dynamic programming fills the values of M.
- Recursion determines order in which table is filled up.
- Think of decomposing problem first (recursion) and then worry about setting up table — this comes naturally from recursion.

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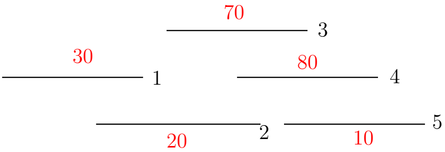
Example



$p(5) = 2$, $p(4) = 1$, $p(3) = 1$, $p(2) = 0$, $p(1) = 0$

Computing Solutions + First Attempt

- Memoization + Recursion/Iteration allows one to compute the optimal value. What about the actual schedule?

$M[0] = 0$
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Computing Implicit Solutions

Observation

Solution can be obtained from $M[]$ in $O(n)$ time, without any additional information

```
findSolution( j )
    if (j = 0) then return empty schedule
    if (v_j + M[p(j)] > M[j - 1]) then
        return findSolution(p(j)) ∪ {j}
    else
        return findSolution(j - 1)
```

Makes $O(n)$ recursive calls, so **findSolution** *runs in $O(n)$ time.*

Computing Implicit Solutions

A generic strategy for computing solutions in dynamic programming:

- Keep track of the *decision* in computing the optimum value of a sub-problem. decision space depends on recursion
- Once the optimum values are computed, go back and use the decision values to compute an optimum solution.

Question: What is the decision in computing $M[i]$?

A: Whether to include i or not.

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Computing Implicit Solutions

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for i = 1 to n do
    M[i] = max(v_i + M[p(i)], M[i − 1])
    if (v_i + M[p(i)] > M[i − 1])then
        Decision[i] = 1 (* 1: i included in solution M[i] *)
    else
        Decision[i] = 0 (* 0: i not included in solution M[i] *)

S = ∅, i = n
while (i > 0) do
    if (Decision[i] = 1) then
        S = S ∪ {i}
        i = p(i)
    else
        i = i − 1
return S
```

Notes

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