Reductions, Recursion and Divide and Conquer

Lecture 5
September 13, 2011
Part I

Reductions and Recursion
Reduction

Reducing problem $A$ to problem $B$:

- Algorithm for $A$ uses algorithm for $B$ as a *black box*
Distinct Elements Problem

Problem: Given an array $A$ of $n$ integers, are there any *duplicates* in $A$?

Naive algorithm:

```plaintext
for $i = 1$ to $n - 1$ do
  for $j = i + 1$ to $n$ do
      return YES

return NO
```

Running time: $O(n^2)$
Distinct Elements Problem

Problem: Given an array $A$ of $n$ integers, are there any duplicates in $A$?

Naive algorithm:

```cpp
for i = 1 to n - 1 do
    for j = i + 1 to n do
        if (A[i] = A[j])
            return YES
    return NO
return NO
```

Running time: $O(n^2)$
Problem  Given an array $A$ of $n$ integers, are there any duplicates in $A$?

Naive algorithm:

```
for $i = 1$ to $n-1$ do
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```

Running time: $O(n^2)$
Problem: Given an array $A$ of $n$ integers, are there any duplicates in $A$?

Naive algorithm:

\[
\begin{array}{l}
\text{for } i = 1 \text{ to } n-1 \text{ do} \\
\quad \text{for } j = i+1 \text{ to } n \text{ do} \\
\quad \quad \text{if } (A[i] = A[j]) \\
\quad \quad \quad \text{return YES} \\
\text{return NO}
\end{array}
\]

Running time: $O(n^2)$
Reduction to Sorting

Sort $A$

for $i = 1$ to $n - 1$ do
  if $(A[i] = A[i + 1])$ then
    return YES

return NO

Running time: $O(n)$ plus time to sort an array of $n$ numbers

Important point: algorithm uses sorting as a black box
Reduction to Sorting

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for $i = 1$ to $n - 1$ do
    if ($A[i] = A[i + 1]$) then
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Running time: $O(n)$ plus time to sort an array of $n$ numbers

Important point: algorithm uses sorting as a black box
Two sides of Reductions

Suppose problem $A$ reduces to problem $B$

- **Positive direction:** Algorithm for $B$ implies an algorithm for $A$
- **Negative direction:** Suppose there is no “efficient” algorithm for $A$ then it implies no efficient algorithm for $B$ (technical condition for reduction time necessary for this)

Example: Distinct Elements reduces to Sorting in $O(n)$ time

- An $O(n \log n)$ time algorithm for Sorting implies an $O(n \log n)$ time algorithm for Distinct Elements problem.
- If there is no $o(n \log n)$ time algorithm for Distinct Elements problem then there is no $o(n \log n)$ time algorithm for Sorting.
Two sides of Reductions

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Example: Distinct Elements reduces to Sorting in $O(n)$ time

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Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction
- reduce problem to a *smaller* instance of *itself*
- self-reduction
  - Problem instance of size $n$ is reduced to one or more instances of size $n - 1$ or less.
  - For termination, problem instances of small size are solved by some other method as *base cases*
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Recursion

- Recursion is a very powerful and fundamental technique
- Basis for several other methods
  - Divide and conquer
  - Dynamic programming
  - Enumeration and branch and bound etc
  - Some classes of greedy algorithms
- Makes proof of correctness easy (via induction)
- Recurrences arise in analysis
Selection Sort

Sort a given array $A[1..n]$ of integers.

Recursive version of Selection sort.

$\text{SelectSort}(A[1..n]):$
\begin{align*}
\text{if } n &= 1 \text{ return } \\
&\text{Find smallest number in } A. \text{ Let } A[i] \text{ be smallest number} \\
&\text{Swap } A[1] \text{ and } A[i] \\
&\text{SelectSort}(A[2..n])
\end{align*}

$T(n)$: time for SelectSort on an $n$ element array.

$T(n) = T(n - 1) + n$ for $n > 1$ and $T(1) = 1$ for $n = 1$

$T(n) = \Theta(n^2)$. 
Selection Sort

Sort a given array $A[1..n]$ of integers.

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Move stack of $n$ disks from peg 0 to peg 2, one disk at a time.

**Rule:** cannot put a larger disk on a smaller disk.

**Question:** what is a strategy and how many moves does it take?
The Tower of Hanoi algorithm; ignore everything but the bottom disk
Recursive Algorithm

\textbf{Hanoi} \( (n, \text{src}, \text{dest}, \text{tmp}) \):

\begin{itemize}
  \item \textbf{if} \( n > 0 \) \textbf{then}
  \item \textbf{Hanoi} \( (n-1, \text{src}, \text{tmp}, \text{dest}) \)
  \item Move disk \( n \) from src to dest
  \item \textbf{Hanoi} \( (n-1, \text{tmp}, \text{dest}, \text{src}) \)
\end{itemize}

\( T(n) \): time to move \( n \) disks via recursive strategy

\[ T(n) = 2T(n-1) + 1 \quad n > 1 \quad \text{and} \quad T(1) = 1 \]
Recursive Algorithm

\textbf{Hanoi}(n, \text{src}, \text{dest}, \text{tmp}): 
  \textbf{if } (n > 0) \textbf{ then} 
  \hspace{1em} \text{Hanoi}(n - 1, \text{src}, \text{tmp}, \text{dest}) 
  \hspace{1em} \text{Move disk } n \text{ from src to dest} 
  \hspace{1em} \text{Hanoi}(n - 1, \text{tmp}, \text{dest}, \text{src}) 

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\(T(n)\): time to move \(n\) disks via recursive strategy

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T(n) = 2T(n - 1) + 1 \quad n > 1 \quad \text{and} \quad T(1) = 1
\]
\begin{align*}
T(n) &= 2T(n - 1) + 1 \\
     &= 2^2 T(n - 2) + 2 + 1 \\
     &= \ldots \\
     &= 2^i T(n - i) + 2^{i-1} + 2^{i-2} + \ldots + 1 \\
     &= \ldots \\
     &= 2^{n-1} T(1) + 2^{n-2} + \ldots + 1 \\
     &= 2^{n-1} + 2^{n-2} + \ldots + 1 \\
     &= (2^n - 1)/(2 - 1) = 2^n - 1
\end{align*}
Non-Recursive Algorithms for Tower of Hanoi

Pegs numbered 0, 1, 2

Non-recursive Algorithm 1:
- Always move smallest disk forward if $n$ is even, backward if $n$ is odd.
- Never move the same disk twice in a row.
- Done when no legal move.

Non-recursive Algorithm 2:
- Let $\rho(n)$ be the smallest integer $k$ such that $n/2^k$ is not an integer. Example: $\rho(40) = 4$, $\rho(18) = 2$.
- In step $i$ move disk $\rho(i)$ forward if $n - i$ is even and backward if $n - i$ is odd.

Moves are exactly same as those of recursive algorithm. Prove by induction.
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Part II

Divide and Conquer
Divide and Conquer Paradigm

Divide and Conquer is a common and useful type of recursion

Approach

- Break problem instance into smaller instances - divide step
- **Recursively** solve problem on smaller instances
- Combine solutions to smaller instances to obtain a solution to the original instance - conquer step

**Question**: Why is this not plain recursion?

- In divide and conquer, each smaller instance is typically at least a constant factor smaller than the original instance which leads to efficient running times.
- There are many examples of this particular type of recursion that it deserves its own treatment.
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- There are many examples of this particular type of recursion that it deserves its own treatment.
Input  Given an array of $n$ elements
Goal  Rearrange them in ascending order
# Merge Sort [von Neumann]

**MergeSort**

1. **Input:** Array $A[1 \ldots n]$ 

2. Divide into subarrays $A[1 \ldots m]$ and $A[m + 1 \ldots n]$, where $m = \lfloor n/2 \rfloor$

3. Recursively **MergeSort** $A[1 \ldots m]$ and $A[m + 1 \ldots n]$

4. Merge the sorted arrays
Merge Sort [von Neumann]

MergeSort

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4. Merge the sorted arrays
Merging Sorted Arrays

- Use a new array \( C \) to store the merged array
- Scan \( A \) and \( B \) from left-to-right, storing elements in \( C \) in order

\[
\begin{align*}
A & \quad G & \quad L & \quad O & \quad R & & & \quad H & \quad I & \quad M & \quad S & \quad T \\
A & \quad G & \quad H & \quad I & \quad L & \quad M & \quad O & \quad R & \quad S & \quad T
\end{align*}
\]

- Merge two arrays using only constantly more extra space (in-place merge sort): doable but complicated and typically impractical
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```
AGLOR
AGHILMORS
```

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- Merge two arrays using only constantly more extra space (in-place merge sort): doable but complicated and typically impractical
Running Time

$T(n)$: time for merge sort to sort an $n$ element array

$$T(n) = T([n/2]) + T([n/2]) + cn$$

What do we want as a solution to the recurrence?

Almost always only an *asymptotically* tight bound. That is we want to know $f(n)$ such that $T(n) = \Theta(f(n))$.

- $T(n) = O(f(n))$ - upper bound
- $T(n) = \Omega(f(n))$ - lower bound
Running Time

\( T(n) \): time for merge sort to sort an \( n \) element array

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T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn
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Solving Recurrences: Some Techniques

- Know some basic math: geometric series, logarithms, exponentials, elementary calculus
- Expand the recurrence and spot a pattern and use simple math
- **Recursion tree method** — imagine the computation as a tree
- **Guess and verify** — useful for proving upper and lower bounds even if not tight bounds

**Albert Einstein**: “Everything should be made as simple as possible, but not simpler.”

Know where to be loose in analysis and where to be tight. Comes with practice, practice, practice!
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MergeSort: \( n \) is a power of 2

- Unroll the recurrence. \( T(n) = 2T(n/2) + cn \)

- Identify a pattern. At the \( i \)th level total work is \( cn \)
- Sum over all levels. The number of levels is \( \log n \). So total is \( cn \log n = O(n \log n) \)
Recursion Trees

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- Identify a pattern. At the $i$th level total work is $cn$
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Recursion Trees

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  ![Recursion Tree Diagram]

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Recursion Trees

An illustrated example...
Recursion Trees

An illustrated example...

Work in each node
Recursion Trees

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\[ \begin{align*}
\log n &=
\left\{ \begin{array}{c}
\frac{cn}{2} + \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} \\
\vdots
\end{array} \right.
\end{align*} \]

\[ = cn \]

\[ \vdots \]

\[ = cn \]
Recursion Trees

An illustrated example...

\[
\begin{align*}
\log n \left\{ \begin{array}{c}
\frac{cn}{2} + \frac{cn}{2} = cn \\
\frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} = cn \\
\vdots \\
\end{array} \right.
\end{align*}
\]

\[= cn \log n = O(n \log n)\]
When \( n \) is not a power of \( 2 \), the running time of \texttt{MergeSort} is expressed as

\[
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn
\]

- \( n_1 = 2^{k-1} < n \leq 2^k = n_2 \) (\( n_1, n_2 \) powers of \( 2 \))
- \( T(n_1) < T(n) \leq T(n_2) \) (Why?)
- \( T(n) = \Theta(n \log n) \) since \( n/2 \leq n_1 < n \leq n_2 \leq 2n \).
When \( n \) is not a power of \( 2 \), the running time of \textbf{MergeSort} is expressed as

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T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + cn
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MergeSort Analysis

When n is not a power of 2

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When $n$ is not a power of 2, the running time of \texttt{MergeSort} is expressed as

$$T(n) = T([n/2]) + T([n/2]) + cn$$

- $n_1 = 2^{k-1} < n \leq 2^k = n_2$ ($n_1, n_2$ powers of 2)
- $T(n_1) < T(n) \leq T(n_2)$ (Why?)
- $T(n) = \Theta(n \log n)$ since $n/2 \leq n_1 < n \leq n_2 \leq 2n$. 

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MergeSort: $n$ is not a power of $2$
\[ T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn \]

**Observation:** For any number \( x \), \( \lfloor x/2 \rfloor + \lceil x/2 \rceil = x \).
MergeSort Analysis

When \( n \) is not a power of 2: Guess and Verify

If \( n \) is power of 2 we saw that \( T(n) = \Theta(n \log n) \).

Can guess that \( T(n) = \Theta(n \log n) \) for all \( n \).

Verify? proof by induction!

Induction Hypothesis: \( T(n) \leq 2cn \log n \) for all \( n \geq 1 \)

Base Case: \( n = 1 \). \( T(1) = 0 \) since no need to do any work and \( 2cn \log n = 0 \) for \( n = 1 \).

Induction Step Assume \( T(k) \leq 2ck \log k \) for all \( k < n \) and prove it for \( k = n \).
MergeSort Analysis

When $n$ is not a power of 2: Guess and Verify

If $n$ is power of $2$ we saw that $T(n) = \Theta(n \log n)$.
Can guess that $T(n) = \Theta(n \log n)$ for all $n$.
Verify? proof by induction!

**Induction Hypothesis:** $T(n) \leq 2cn \log n$ for all $n \geq 1$

**Base Case:** $n = 1$. $T(1) = 0$ since no need to do any work and $2cn \log n = 0$ for $n = 1$.

**Induction Step** Assume $T(k) \leq 2ck \log k$ for all $k < n$ and prove it for $k = n$. 
We have

\[
T(n) = T([n/2]) + T([n/2]) + cn
\]
\[
\leq 2c\lfloor n/2 \rfloor \log \lfloor n/2 \rfloor + 2c\lceil n/2 \rceil \log \lceil n/2 \rceil + cn \quad \text{(by induction)}
\]
\[
\leq 2c\lceil n/2 \rceil \log \lceil n/2 \rceil + 2c\lfloor n/2 \rfloor \log \lceil n/2 \rceil + cn
\]
\[
\leq 2c(\lceil n/2 \rceil + \lfloor n/2 \rfloor) \log \lfloor n/2 \rfloor + cn
\]
\[
\leq 2cn \log \lfloor n/2 \rfloor + cn
\]
\[
\leq 2cn \log (2n/3) + cn \quad \text{(since \( \lceil n/2 \rceil \leq 2n/3 \) for all \( n \geq 2 \))}
\]
\[
\leq 2cn \log n + cn(1 - 2 \log 3/2)
\]
\[
\leq 2cn \log n + cn(\log 2 - \log 9/4)
\]
\[
\leq 2cn \log n
\]
The math worked out like magic!

Why was $2cn \log n$ chosen instead of say $4cn \log n$?

- Do not know upfront what constant to choose.
- Instead assume that $T(n) \leq \alpha cn \log n$ for some constant $\alpha$. $\alpha$ will be fixed later.
- Need to prove that for $\alpha$ large enough the algebra go through.
- In our case... need $\alpha$ such that $\alpha \log 3/2 > 1$.
- Typically, do the algebra with $\alpha$ and then show that it works... ... if $\alpha$ is chosen to be sufficiently large constant.
Guess and Verify

What happens if the guess is wrong?

- Guessed that the solution to the **MergeSort** recurrence is \( T(n) = O(n) \).
- Try to prove by induction that \( T(n) \leq \alpha cn \) for some const’ \( \alpha \).

**Induction Step:** attempt

\[
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn \\
\leq \alpha c \lfloor n/2 \rfloor + \alpha c \lceil n/2 \rceil + cn \\
\leq \alpha cn + cn \\
\leq (\alpha + 1)cn
\]

But need to show that \( T(n) \leq \alpha cn \! \)

- So guess does not work for any constant \( \alpha \). Suggest that our guess is incorrect.
Selection Sort vs Merge Sort

- Selection Sort spends $O(n)$ work to reduce problem from $n$ to $n - 1$ leading to $O(n^2)$ running time.
- Merge Sort spends $O(n)$ time after reducing problem to two instances of size $n/2$ each. Running time is $O(n \log n)$

**Question**: Merge Sort splits into 2 (roughly) equal sized arrays. Can we do better by splitting into more than 2 arrays? Say $k$ arrays of size $n/k$ each?
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Quick Sort

Quick Sort [Hoare]

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself. Linear scan of array does it. Time is $O(n)$
3. Recursively sort the subarrays, and concatenate them.

Example:

- array: 16, 12, 14, 20, 5, 3, 18, 19, 1
- pivot: 16
- split into 12, 14, 5, 3, 1 and 20, 19, 18 and recursively sort
- put them together with pivot in middle
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Example:
- array: $16, 12, 14, 20, 5, 3, 18, 19, 1$
- pivot: $16$
- split into $12, 14, 5, 3, 1$ and $20, 19, 18$ and recursively sort
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Time Analysis

- Let $k$ be the rank of the chosen pivot. Then,
  \[ T(n) = T(k - 1) + T(n - k) + O(n) \]

- If $k = \lceil n/2 \rceil$ then
  \[ T(n) = T(\lceil n/2 \rceil - 1) + T(\lfloor n/2 \rfloor) + O(n) \leq 2T(n/2) + O(n). \]
  Then, \[ T(n) = O(n \log n). \]

- Theoretically, median can be found in linear time.

- Typically, pivot is the first or last element of array. Then,

  \[ T(n) = \max_{1 \leq k \leq n} (T(k - 1) + T(n - k) + O(n)) \]

In the worst case \[ T(n) = T(n - 1) + O(n), \] which means \[ T(n) = O(n^2). \] Happens if array is already sorted and pivot is always first element.
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Part III

Fast Multiplication
Multiplying Numbers

Problem Given two \( n \)-digit numbers \( x \) and \( y \), compute their product.

Grade School Multiplication

Compute “partial product” by multiplying each digit of \( y \) with \( x \) and adding the partial products.

\[
\begin{array}{c}
3141 \\
\times 2718 \\
\hline
25128 \\
3141 \\
21987 \\
6282 \\
\hline
8537238
\end{array}
\]
Time Analysis of Grade School Multiplication

- Each partial product: $\Theta(n)$
- Number of partial products: $\Theta(n)$
- Addition of partial products: $\Theta(n^2)$
- Total time: $\Theta(n^2)$
A Trick of Gauss

Carl Friedrich Gauss: 1777–1855 “Prince of Mathematicians”

Observation: Multiply two complex numbers: \((a + bi)\) and \((c + di)\)

\[(a + bi)(c + di) = ac - bd + (ad + bc)i\]

How many multiplications do we need?

Only 3! If we do extra additions and subtractions.
Compute \(ac, bd, (a + b)(c + d)\). Then
\[(ad + bc) = (a + b)(c + d) - ac - bd\]
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Compute \(ac, bd, (a + b)(c + d)\). Then

\[
(ad + bc) = (a + b)(c + d) - ac - bd
\]
Divide and Conquer

Assume \( n \) is a power of 2 for simplicity and numbers are in decimal.

- \( x = x_{n-1}x_{n-2} \cdots x_0 \) and \( y = y_{n-1}y_{n-2} \cdots y_0 \)
- \( x = 10^{n/2}x_L + x_R \) where \( x_L = x_{n-1} \cdots x_{n/2} \) and \( x_R = x_{n/2-1} \cdots x_0 \)
- \( y = 10^{n/2}y_L + y_R \) where \( y_L = y_{n-1} \cdots y_{n/2} \) and \( y_R = y_{n/2-1} \cdots y_0 \)

Therefore

\[
x y = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) = 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R
\]
Example

\[
1234 \times 5678 = (100 \times 12 + 34) \times (100 \times 56 + 78) \\
= 10000 \times 12 \times 56 \\
+ 100 \times (12 \times 78 + 34 \times 56) \\
+ 34 \times 78
\]
Time Analysis

\[ \begin{align*}
xy &= (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \\
&= 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R
\end{align*} \]

4 recursive multiplications of number of size \( n/2 \) each plus 4 additions and left shifts (adding enough 0’s to the right)

\[ T(n) = 4T(n/2) + O(n) \quad T(1) = O(1) \]

\[ T(n) = \Theta(n^2) \]. No better than grade school multiplication!

Can we invoke Gauss’s trick here?
Time Analysis

\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \]
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Improving the Running Time

\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) = 10^n x_L y_L + 10^{n/2} (x_L y_R + x_R y_L) + x_R y_R \]

Gauss trick: \( x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R \)

Recursively compute only \( x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R) \).

**Time Analysis**

Running time is given by

\[ T(n) = 3T(n/2) + O(n) \quad T(1) = O(1) \]

which means \( T(n) = O(n^{\log_2 3}) = O(n^{1.585}) \)
Improving the Running Time

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State of the Art

Schönhage-Strassen 1971: $O(n \log n \log \log n)$ time using Fast-Fourier-Transform (FFT)

Martin Fürer 2007: $O(n \log n 2^{O(\log^* n)})$ time

Conjecture

There is an $O(n \log n)$ time algorithm.
Analyzing the Recurrences

- **Basic divide and conquer:** \( T(n) = 4T(n/2) + O(n) \), \( T(1) = 1 \). **Claim:** \( T(n) = \Theta(n^2) \).
- **Saving a multiplication:** \( T(n) = 3T(n/2) + O(n) \), \( T(1) = 1 \). **Claim:** \( T(n) = \Theta(n^{1+\log 1.5}) \)

Use recursion tree method:

- In both cases, depth of recursion \( L = \log n \).
- Work at depth \( i \) is \( 4^i n/2^i \) and \( 3^i n/2^i \) respectively: number of children at depth \( i \) times the work at each child.
- Total work is therefore \( n \sum_{i=0}^{L} 2^i \) and \( n \sum_{i=0}^{L} (3/2)^i \) respectively.
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