Reductions, Recursion and Divide and Conquer

Lecture 5

September 13, 2011

Part I

Reductions and Recursion

Reduction

Reducing problem $\boldsymbol{A}$ to problem $\boldsymbol{B}$:

- Algorithm for $\boldsymbol{A}$ uses algorithm for $\boldsymbol{B}$ as a *black box*

Distinct Elements Problem

Problem: Given an array A of n integers, are there any *duplicates* in A?

Naive algorithm:

```
for i = 1 to n − 1 do
    for j = i + 1 to n do
        if (A[i] = A[j])
            return YES
return NO
```

Running time: $O(n^2)$

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Reduction to Sorting

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Sort A
for i = 1 to n − 1 do
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Running time: $O(n)$ plus time to sort an array of n numbers

Important point: algorithm uses sorting as a black box

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Two sides of Reductions

Suppose problem A reduces to problem B

- Positive direction: Algorithm for B implies an algorithm for A
- Negative direction: Suppose there is no "efficient" algorithm for A then it implies no efficient algorithm for B (technical condition for reduction time necessary for this)

Example: Distinct Elements reduces to Sorting in $O(n)$ time

- An $O(n \log n)$ time algorithm for Sorting implies an $O(n \log n)$ time algorithm for Distinct Elements problem.
- If there is *no* $o(n \log n)$ time algorithm for Distinct Elements problem then there is *no* $o(n \log n)$ time algorithm for Sorting.

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Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction

- reduce problem to a *smaller* instance of *itself*
- self-reduction
- Problem instance of size n is reduced to one or more instances of size $n-1$ or less.
- For termination, problem instances of small size are solved by some other method as *base cases*

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Recursion

- Recursion is a very powerful and fundamental technique
- Basis for several other methods
  - Divide and conquer
  - Dynamic programming
  - Enumeration and branch and bound etc
  - Some classes of greedy algorithms
- Makes proof of correctness easy (via induction)
- Recurrences arise in analysis

Selection Sort

Sort a given array $A[1..n]$ of integers.

Recursive version of Selection sort.

SelectSort($A[1..n]$):
    if $n = 1$ **return**
    Find smallest number in A. Let $A[i]$ be smallest number
    Swap $A[1]$ and $A[i]$
    SelectSort($A[2..n]$)

$T(n)$: time for **SelectSort** on an n element array.

$T(n) = T(n-1) + n$ for $n > 1$ and $T(1) = 1$ for $n = 1$

$T(n) = \Theta(n^2)$.

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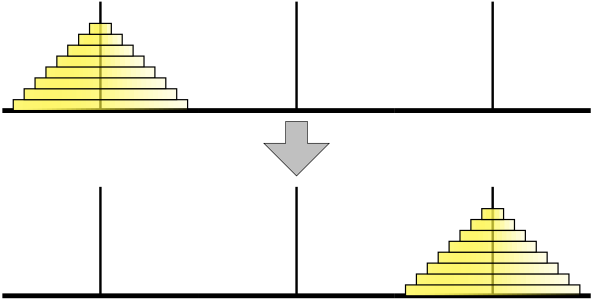
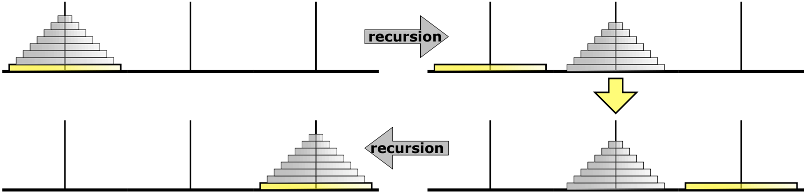
Tower of Hanoi



The Tower of Hanoi puzzle

Move stack of n disks from peg $\mathbf{0}$ to peg $\mathbf{2}$, one disk at a time.
Rule: cannot put a larger disk on a smaller disk.
Question: what is a strategy and how many moves does it take?

Tower of Hanoi via Recursion



The Tower of Hanoi algorithm; ignore everything but the bottom disk

Recursive Algorithm

```
Hanoi(n, src, dest, tmp):
    if (n > 0) then
        Hanoi(n − 1, src, tmp, dest)
        Move disk n from src to dest
        Hanoi(n − 1, tmp, dest, src)
```

$T(n)$: time to move n disks via recursive strategy

$$T(n) = 2T(n-1) + 1 \qquad n > 1 \qquad \text{and } T(1) = 1$$

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Analysis

$$
\begin{aligned}
T(n) &= 2T(n-1)+1\\
&= 2^2T(n-2)+2+1\\
&= \ldots\\
&= 2^iT(n-i)+2^{i-1}+2^{i-2}+\ldots+1\\
&= \ldots\\
&= 2^{n-1}T(1)+2^{n-2}+\ldots+1\\
&= 2^{n-1}+2^{n-2}+\ldots+1\\
&= (2^n-1)/(2-1)=2^n-1
\end{aligned}
$$

Non-Recursive Algorithms for Tower of Hanoi

Pegs numbered $\mathbf{0, 1, 2}$

Non-recursive Algorithm 1:

- Always move smallest disk forward if n is even, backward if n is odd.
- Never move the same disk twice in a row.
- Done when no legal move.

Non-recursive Algorithm 2:

- Let $\rho(n)$ be the smallest integer k such that $n/2^k$ is *not* an integer. Example: $\rho(40) = 4$, $\rho(18) = 2$.
- In step i move disk $\rho(i)$ forward if $n - i$ is even and backward if $n - i$ is odd.

Moves are exactly same as those of recursive algorithm. Prove by induction.

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Part II

Divide and Conquer

Divide and Conquer Paradigm

Divide and Conquer is a common and useful type of recursion

Approach

- Break problem instance into smaller instances - divide step
- Recursively solve problem on smaller instances
- Combine solutions to smaller instances to obtain a solution to the original instance - conquer step

Question: Why is this not plain recursion?

- In divide and conquer, each smaller instance is typically at least a constant factor smaller than the original instance which leads to efficient running times.
- There are many examples of this particular type of recursion that it deserves its own treatment.

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Sorting

Input Given an array of n elements

Goal Rearrange them in ascending order

Merge Sort [von Neumann]

MergeSort

1. Input: Array $A[1 \ldots n]$

   $A\ L\ G\ O\ R\ I\ T\ H\ M\ S$

2. Divide into subarrays $A[1 \ldots m]$ and $A[m+1 \ldots n]$, where $m = \lfloor n/2 \rfloor$

   $A\ L\ G\ O\ R \quad I\ T\ H\ M\ S$

3. Recursively MergeSort $A[1 \ldots m]$ and $A[m+1 \ldots n]$

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4. Merge the sorted arrays

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Merging Sorted Arrays

- Use a new array C to store the merged array
- Scan A and B from left-to-right, storing elements in C in order

$$A\ G\ L\ O\ R \qquad H\ I\ M\ S\ T$$
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Running Time

$T(n)$: time for merge sort to sort an n element array

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

What do we want as a solution to the recurrence?

Almost always only an *asymptotically* tight bound. That is we want to know $f(n)$ such that $T(n) = \Theta(f(n))$.

- $T(n) = O(f(n))$ - upper bound
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Solving Recurrences: Some Techniques

- Know some basic math: geometric series, logarithms, exponentials, elementary calculus
- Expand the recurrence and spot a pattern and use simple math
- Recursion tree method — imagine the computation as a tree
- Guess and verify — useful for proving upper and lower bounds even if not tight bounds

Albert Einstein: "Everything should be made as simple as possible, but not simpler."

Know where to be loose in analysis and where to be tight. Comes with practice, practice, practice!

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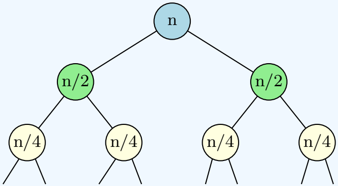
Recursion Trees

MergeSort: n is a power of 2

- Unroll the recurrence. $T(n) = 2T(n/2) + cn$



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- Sum over all levels. The number of levels is $\log n$. So total is $cn \log n = O(n \log n)$

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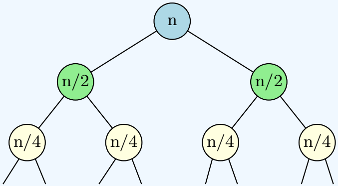

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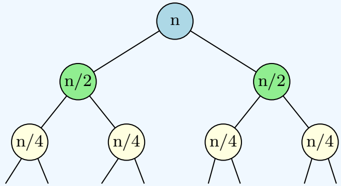

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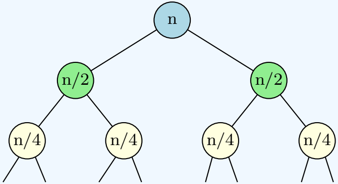

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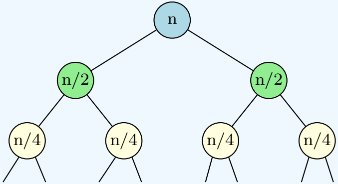

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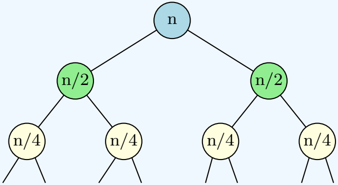
An illustrated example...



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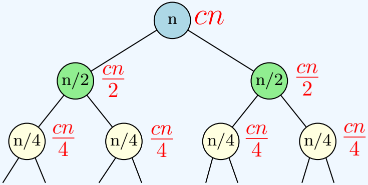

Work in each node

Recursion Trees

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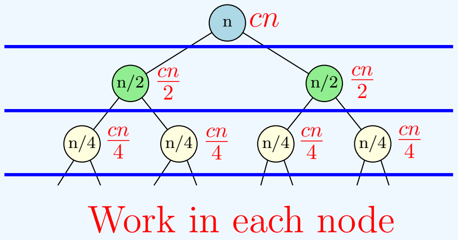

Recursion Trees

An illustrated example...

$$\log n \left\{ \begin{array}{cc} cn & = cn \\ \hline \frac{cn}{2} \quad + \quad \frac{cn}{2} & = cn \\ \hline \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} & = cn \\ \hline \vdots & \vdots \\ \hline & = cn \end{array} \right.$$

Recursion Trees

An illustrated example...

$$
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cn \\
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\hline
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\hline
\vdots \\
\hline
\end{array}
\right.
\quad
\begin{array}{l}
= cn \\
\quad + \\
= cn \\
\quad + \\
= cn \\
\quad \vdots \\
= cn
\end{array}
$$

$$= cn \log n = O(n \log n)$$

MergeSort Analysis

When n is not a power of 2

- When n is not a power of 2, the running time of **MergeSort** is expressed as

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

- $n_1 = 2^{k-1} < n \le 2^k = n_2$ (n_1, n_2 powers of 2)
- $T(n_1) < T(n) \le T(n_2)$ (Why?)
- $T(n) = \Theta(n \log n)$ since $n/2 \le n_1 < n \le n_2 \le 2n$.

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- When n is not a power of 2, the running time of **MergeSort** is expressed as

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

- $n_1 = 2^{k-1} < n \leq 2^k = n_2$ (n_1, n_2 powers of 2)
- $T(n_1) < T(n) \leq T(n_2)$ (Why?)
- $T(n) = \Theta(n \log n)$ since $n/2 \leq n_1 < n \leq n_2 \leq 2n$.

Recursion Trees

MergeSort: $\mathbf{n}$ is not a power of $\mathbf{2}$

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

Observation: For any number x, $\lfloor x/2 \rfloor + \lceil x/2 \rceil = x$.

MergeSort Analysis

When n is not a power of 2: Guess and Verify

If n is power of 2 we saw that $T(n) = \Theta(n \log n)$.
Can *guess* that $T(n) = \Theta(n \log n)$ for all n.
Verify? proof by induction!

Induction Hypothesis: $T(n) \leq 2cn \log n$ for all $n \geq 1$
Base Case: $n = 1$. $T(1) = 0$ since no need to do any work and $2cn \log n = 0$ for $n = 1$.
Induction Step Assume $T(k) \leq 2ck \log k$ for all $k < n$ and prove it for $k = n$.

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Induction Step

We have

$$
\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn \\
&\leq 2c\lfloor n/2 \rfloor \log \lfloor n/2 \rfloor + 2c\lceil n/2 \rceil \log \lceil n/2 \rceil + cn \quad \text{(by induct} \\
&\leq 2c\lfloor n/2 \rfloor \log \lceil n/2 \rceil + 2c\lceil n/2 \rceil \log \lceil n/2 \rceil + cn \\
&\leq 2c(\lfloor n/2 \rfloor + \lceil n/2 \rceil) \log \lceil n/2 \rceil + cn \\
&\leq 2cn \log \lceil n/2 \rceil + cn \\
&\leq 2cn \log (2n/3) + cn \quad \text{(since } \lceil n/2 \rceil \leq 2n/3 \text{ for all } n \geq 2\text{)} \\
&\leq 2cn \log n + cn(1 - 2 \log 3/2) \\
&\leq 2cn \log n + cn(\log 2 - \log 9/4) \\
&\leq 2cn \log n
\end{aligned}
$$

Guess and Verify

The math worked out like magic!
Why was $2cn \log n$ chosen instead of say $4cn \log n$?

- Do not know upfront what constant to choose.
- Instead assume that $T(n) \leq \alpha cn \log n$ for some constant α. α will be fixed later.
- Need to prove that for α large enough the algebra go through.
- In our case... need α such that $\alpha \log 3/2 > 1$.
- Typically, do the algebra with α and then show that it works... ... if α is chosen to be sufficiently large constant.

Guess and Verify

What happens if the guess is wrong?

- Guessed that the solution to the **MergeSort** recurrence is $T(n) = O(n)$.
- Try to prove by induction that $T(n) \le \alpha cn$ for some const' α.
  Induction Step: attempt

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn \\ &\le \alpha c \lfloor n/2 \rfloor + \alpha c \lceil n/2 \rceil + cn \\ &\le \alpha cn + cn \\ &\le (\alpha + 1)cn \end{aligned}$$

  But need to show that $T(n) \le \alpha cn$!
- So guess does not work for any constant α. Suggest that our guess is incorrect.

Selection Sort vs Merge Sort

- Selection Sort spends $O(n)$ work to reduce problem from n to $n - 1$ leading to $O(n^2)$ running time.
- Merge Sort spends $O(n)$ time *after* reducing problem to two instances of size $n/2$ each. Running time is $O(n \log n)$

Question: Merge Sort splits into 2 (roughly) equal sized arrays. Can we do better by splitting into more than 2 arrays? Say k arrays of size n/k each?

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Quick Sort

Quick Sort [Hoare]

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself. Linear scan of array does it. Time is $O(n)$
3. Recursively sort the subarrays, and concatenate them.

Example:

- array: 16, 12, 14, 20, 5, 3, 18, 19, 1
- pivot: 16
- split into 12, 14, 5, 3, 1 and 20, 19, 18 and recursively sort
- put them together with pivot in middle

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Time Analysis

- Let k be the rank of the chosen pivot. Then, $T(n) = T(k-1) + T(n-k) + O(n)$
- If $k = \lceil n/2 \rceil$ then $T(n) = T(\lceil n/2 \rceil - 1) + T(\lfloor n/2 \rfloor) + O(n) \leq 2T(n/2) + O(n)$. Then, $T(n) = O(n \log n)$.
  - Theoretically, median can be found in linear time.
- Typically, pivot is the first or last element of array. Then,

$$T(n) = \max_{1 \leq k \leq n} \left(T(k-1) + T(n-k) + O(n) \right)$$

In the worst case $T(n) = T(n-1) + O(n)$, which means $T(n) = O(n^2)$. Happens if array is already sorted and pivot is always first element.

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Part III

Fast Multiplication

Multiplying Numbers

Problem Given two n-digit numbers x and y, compute their product.

Grade School Multiplication

Compute "partial product" by multiplying each digit of y with x and adding the partial products.

$$\begin{array}{r}
\mathbf{3141} \\
\underline{\times\mathbf{2718}} \\
\mathbf{25128} \\
\mathbf{3141} \\
\mathbf{21987} \\
\underline{\mathbf{6282}} \\
\mathbf{8537238}
\end{array}$$

Time Analysis of Grade School Multiplication

- Each partial product: $\Theta(n)$
- Number of partial products: $\Theta(n)$
- Addition of partial products: $\Theta(n^2)$
- Total time: $\Theta(n^2)$

A Trick of Gauss

Carl Fridrich Gauss: 1777–1855 "Prince of Mathematicians"

Observation: Multiply two complex numbers: $(a + bi)$ and $(c + di)$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

How many multiplications do we need?

Only 3! If we do extra additions and subtractions.
Compute ac, bd, $(a + b)(c + d)$. Then
$(ad + bc) = (a + b)(c + d) - ac - bd$

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Compute $ac, bd, (a + b)(c + d)$. Then
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Divide and Conquer

Assume n is a power of 2 for simplicity and numbers are in decimal.

- $x = x_{n-1}x_{n-2}\ldots x_0$ and $y = y_{n-1}y_{n-2}\ldots y_0$
- $x = 10^{n/2}x_L + x_R$ where $x_L = x_{n-1}\ldots x_{n/2}$ and $x_R = x_{n/2-1}\ldots x_0$
- $y = 10^{n/2}y_L + y_R$ where $y_L = y_{n-1}\ldots y_{n/2}$ and $y_R = y_{n/2-1}\ldots y_0$

Therefore

$$\begin{aligned} xy &= (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \\ &= 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R \end{aligned}$$

Example

$$
\begin{aligned}
1234 \times 5678 &= (100 \times 12 + 34) \times (100 \times 56 + 78) \\
&= 10000 \times 12 \times 56 \\
&\quad +100 \times (12 \times 78 + 34 \times 56) \\
&\quad +34 \times 78
\end{aligned}
$$

Time Analysis

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4 recursive multiplications of number of size $n/2$ each plus 4 additions and left shifts (adding enough 0's to the right)

$$T(n) = 4T(n/2) + O(n) \qquad T(1) = O(1)$$

$T(n) = \Theta(n^2)$. No better than grade school multiplication!

Can we invoke Gauss's trick here?

Time Analysis

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Gauss trick: $x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$

Recursively compute only $x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R)$.

Time Analysis

Running time is given by

$$T(n) = 3T(n/2) + O(n) \qquad T(1) = O(1)$$

which means $T(n) = O(n^{\log_2 3}) = O(n^{1.585})$

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State of the Art

Schönhage-Strassen 1971: $O(n \log n \log \log n)$ time using Fast-Fourier-Transform (FFT)

Martin Fürer 2007: $O(n \log n 2^{O(\log^* n)})$ time

Conjecture

There is an $O(n \log n)$ time algorithm.

Analyzing the Recurrences

- Basic divide and conquer: $T(n) = 4T(n/2) + O(n)$, $T(1) = 1$. **Claim:** $T(n) = \Theta(n^2)$.
- Saving a multiplication: $T(n) = 3T(n/2) + O(n)$, $T(1) = 1$. **Claim:** $T(n) = \Theta(n^{1+\log 1.5})$

Use recursion tree method:

- In both cases, depth of recursion $L = \log n$.
- Work at depth i is $4^i n/2^i$ and $3^i n/2^i$ respectively: number of children at depth i times the work at each child
- Total work is therefore $n\sum_{i=0}^{L} 2^i$ and $n\sum_{i=0}^{L}(3/2)^i$ respectively.

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Recursion tree analysis

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