Chapter 3

Breadth First Search, Dijkstra's Algorithm for Shortest Paths

CS 473: Fundamental Algorithms, Fall 2011
August 30, 2011

3.1 Breadth First Search

3.1.0.1 Breadth First Search (BFS)

Overview

1. **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a ***queue***.

2. It processes the vertices in the graph in the order of their shortest distance from the vertex s (the start vertex).

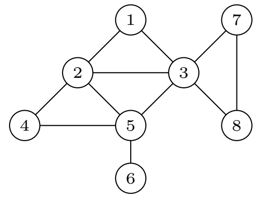
As such...

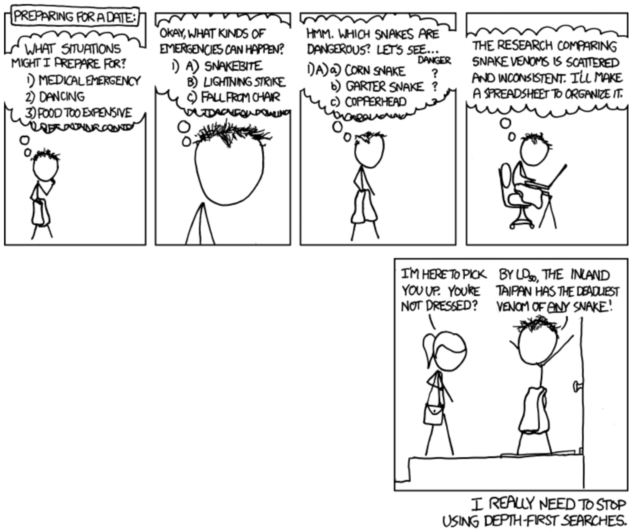
(A) **DFS** good for exploring graph structure
(B) **BFS** good for exploring *distances*

3.1.0.2 Queue Data Structure

Queues

A ***queue*** is a list of elements which supports the operations:
(A) **enqueue**: Adds an element to the end of the list
(B) **dequeue**: Removes an element from the front of the list
Elements are extracted in ***first-in first-out (FIFO)*** order, i.e., elements are picked in the order in which they were inserted.





3.1.0.3 BFS Algorithm

Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

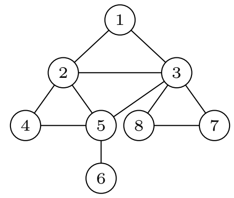
```
BFS(s)
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u, v) to T
                Mark v as visited and enq(v)
```

Proposition 3.1.1 **BFS**(s) *runs in* $O(n + m)$ *time.*

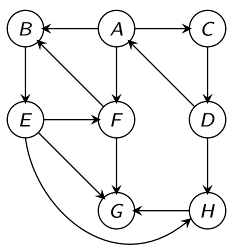
3.1.0.4 BFS: An Example in Undirected Graphs

1.	[1]	4.	[4,5,7,8]	7.	[8,6]
2.	[2,3]	5.	[5,7,8]	8.	[6]
3.	[3,4,5]	6.	[7,8,6]	9.	[]

BFS tree is the set of black edges.



3.1.0.5 BFS: An Example in Directed Graphs



3.1.0.6 BFS with Distance

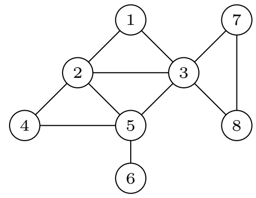
BFS(s)
    Mark all vertices as unvisited *and for each* v *set* $\mathrm{dist}(v) = \infty$
    Initialize search tree T to be empty
    Mark vertex s as visited *and set* $\mathrm{dist}(s) = 0$
    set Q to be the empty queue
    enq(s)
    while Q is nonempty **do**
        $u =$ **deq**(Q)
        for each vertex $v \in \mathrm{Adj}(u)$ **do**
            if v is not visited **do**
                add edge (u, v) to T
                Mark v as visited, **enq**(v)
                and set $\mathrm{dist}(v) = \mathrm{dist}(u) + 1$

3.1.0.7 Properties of BFS: Undirected Graphs

Proposition 3.1.2 *The following properties hold upon* termination *of* **BFS**(s)

1. *The search tree contains exactly the set of vertices in the connected component of* s.
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v.
3. *For every vertex* u, $\mathrm{dist}(u)$ *is indeed the length of shortest path from* s *to* u.
4. *If* u, v *are in connected component of* s *and* $e = \{u, v\}$ *is an edge of* G, *then either* e *is an edge in the search tree, or* $|\mathrm{dist}(u) - \mathrm{dist}(v)| \leq 1$.

Proof: Exercise. ∎



3.1.0.8 Properties of BFS: Directed Graphs

Proposition 3.1.3 *The following properties hold upon termination of* **BFS**(s)*:*

1. *The search tree contains exactly the set of vertices reachable from* s
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v
3. *For every vertex* u*,* $\mathrm{dist}(u)$ *is indeed the length of shortest path from* s *to* u
4. *If* u *is reachable from* s *and* $e = (u, v)$ *is an edge of* G*, then either* e *is an edge in the search tree, or* $\mathrm{dist}(v) - \mathrm{dist}(u) \leq 1$*. Not necessarily the case that* $\mathrm{dist}(u) - \mathrm{dist}(v) \leq 1$*.*

Proof: Exercise. ∎

3.1.0.9 BFS with Layers

```
BFSLayers(s):
Mark all vertices as unvisited and initialize T to be empty
Mark s as visited and set L_0 = {s}
i = 0
while L_i is not empty do
        initialize L_{i+1} to be an empty list
        for each u in L_i do
            for each edge (u,v) ∈ Adj(u) do
            if v is not visited
                    mark v as visited
                    add (u,v) to tree T
                    add v to L_{i+1}
        i = i + 1
```

Running time: $O(n + m)$

3.1.0.10 Example
3.1.0.11 BFS with Layers: Properties

Proposition 3.1.4 *The following properties hold on termination of* **BFSLayers**(s)*.*
(A) **BFSLayers**(s) *outputs a* **BFS** *tree*
(B) L_i *is the set of vertices at distance exactly* i *from* s
(C) If G *is undirected, each edge* $e = \{u, v\}$ *is one of three types:*
  (A) **tree** *edge between two* consecutive *layers*

(B) non-tree **forward/backward** *edge between two consecutive layers*

(C) non-tree **cross-edge** *with both u, v in same layer*

(D) $\implies$ Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.

3.1.1 BFS with Layers: Properties

3.1.1.1 For directed graphs

Proposition 3.1.5 *The following properties hold on termination of* **BFSLayers**(s)*, if G is directed.*

For each edge $e = (u, v)$ is one of four types:

(A) a **tree** *edge between consecutive layers, $u \in L_i, v \in L_{i+1}$ for some $i \geq 0$*

(B) a non-tree **forward** *edge between consecutive layers*

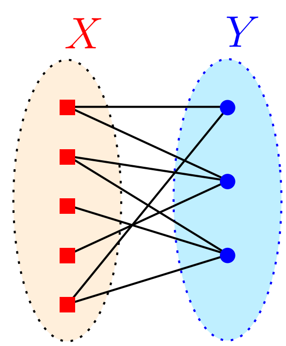
(C) a non-tree **backward** *edge*

(D) a **cross-edge** *with both u, v in same layer*

3.2 Bipartite Graphs and an application of BFS

3.2.0.2 Bipartite Graphs

Definition 3.2.1 (Bipartite Graph) *Undirected graph $G = (V, E)$ is a* **bipartite graph** *if V can be partitioned into X and Y s.t. all edges in E are between X and Y.*



3.2.0.3 Bipartite Graph Characterization

Question
When is a graph bipartite?

Proposition 3.2.2 *Every tree is a bipartite graph.*

Proof: Root tree T at some node r. Let L_i be all nodes at level i, that is, L_i is all nodes at distance i from root r. Now define X to be all nodes at even levels and Y to be all nodes at odd level. Only edges in T are between levels. ■

Proposition 3.2.3 *An odd length cycle is not bipartite.*

3.2.0.4 Odd Cycles are not Bipartite

Proposition 3.2.4 *An odd length cycle is not bipartite.*

Proof: Let $C = u_1, u_2, \ldots, u_{2k+1}, u_1$ be an odd cycle. Suppose C is a bipartite graph and let X, Y be the partition. Without loss of generality $u_1 \in X$. Implies $u_2 \in Y$. Implies $u_3 \in X$. Inductively, $u_i \in X$ if i is odd $u_i \in Y$ if i is even. But $\{u_1, u_{2k+1}\}$ is an edge and both belong to X! ■

3.2.0.5 Subgraphs

Definition 3.2.5 *Given a graph $G = (V, E)$ a* **subgraph** *of G is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.*

Proposition 3.2.6 *If G is bipartite then any subgraph H of G is also bipartite.*

Proposition 3.2.7 *A graph G is not bipartite if G has an odd cycle C as a subgraph.*

Proof: If G is bipartite then since C is a subgraph, C is also bipartite (by above proposition). However, C is not bipartite! ■

3.2.0.6 Bipartite Graph Characterization

Theorem 3.2.8 *A graph G is bipartite if and only if it has no odd length cycle as subgraph.*

Proof: *Only If:* G has an odd cycle implies G is not bipartite.
If: G has no odd length cycle. Assume without loss of generality that G is connected.
(A) Pick u arbitrarily and do **BFS**(u)
(B) $X = \cup_{i \text{ is even}} L_i$ and $Y = \cup_{i \text{ is odd}} L_i$
(C) **Claim:** X and Y is a valid partition if G has no odd length cycle.

■

3.2.0.7 Proof of Claim

Claim 3.2.9 *In* **BFS**(u) *if $a, b \in L_i$ and (a, b) is an edge then there is an odd length cycle containing (a, b).*

Proof: Let v be least common ancestor of a, b in **BFS** tree T.
v is in some level $j < i$ (could be u itself).
Path from $v \rightsquigarrow a$ in T is of length $j - i$.
Path from $v \rightsquigarrow b$ in T is of length $j - i$.
These two paths plus (a, b) forms an odd cycle of length $2(j - i) + 1$. ■

3.2.0.8 Another tidbit

Corollary 3.2.10 *There is an $O(n+m)$ time algorithm to check if G is bipartite and output an odd cycle if it is not.*

3.3 Shortest Paths and Dijkstra's Algorithm

3.3.0.9 Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

(A) Given nodes s, t find shortest path from s to t.
(B) Given node s find shortest path from s to all other nodes.
(C) Find shortest paths for all pairs of nodes.

Many applications!

3.3.1 Single-Source Shortest Paths:

3.3.1.1 Non-Negative Edge Lengths

Single-Source Shortest Path Problems

(A) *Input*: A (undirected or directed) graph $G = (V, E)$ with *non-negative* edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.
(B) Given nodes s, t find shortest path from s to t.
(C) Given node s find shortest path from s to all other nodes.

(A) Restrict attention to directed graphs
(B) Undirected graph problem can be reduced to directed graph problem - how?
    (A) Given undirected graph G, create a new directed graph G' by replacing each edge $\{u, v\}$ in G by (u, v) and (v, u) in G'.
    (B) set $\ell(u, v) = \ell(v, u) = \ell(\{u, v\})$
    (C) Exercise: show reduction works

3.3.1.2 Single-Source Shortest Paths via BFS

Special case: All edge lengths are 1.

(A) Run **BFS**(s) to get shortest path distances from s to all other nodes.
(B) $O(m + n)$ time algorithm.

Special case: Suppose $\ell(e)$ is an integer for all e?
Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e

Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. **BFS** takes $O(mL + n)$ time. Not efficient if L is large.

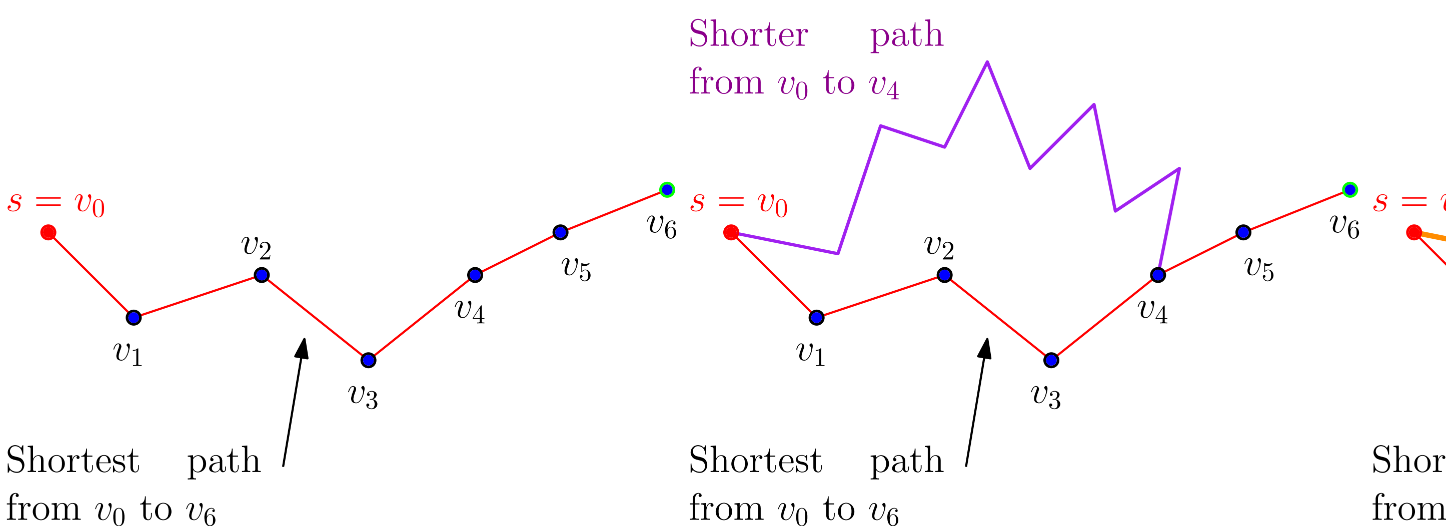
3.3.1.3 Towards an algorithm

Why does **BFS** work?
BFS(s) explores nodes in increasing distance from s

Lemma 3.3.1 *Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ is a shortest path from s to v_k then for $1 \le i < k$:*
(A) $s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ *is a shortest path from s to v_i*
(B) $\mathrm{dist}(s, v_i) \le \mathrm{dist}(s, v_k)$.

Proof: Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \to v_1 \to \ldots \to v_i$. Then P' concatenated with $v_i \to v_{i+1} \ldots \to v_k$ contains a strictly shorter path to v_k than $s = v_0 \to v_1 \ldots \to v_k$. ■

3.3.1.4 A proof by picture



3.3.1.5 A Basic Strategy

Explore vertices in increasing order of distance from s:

(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

Initialize for each node v, $\text{dist}(s, v) = \infty$
Initialize $S = \emptyset$,
for $i = 1$ to $|V|$ **do**
    (Invariant: S contains the $i-1$ closest nodes to s *)*
    Among nodes in $V \setminus S$, find the node v that is the
        ith closest to s
    Update $\text{dist}(s, v)$
    $S = S \cup \{v\}$

How can we implement the step in the for loop?

3.3.1.6 Finding the ith closest node

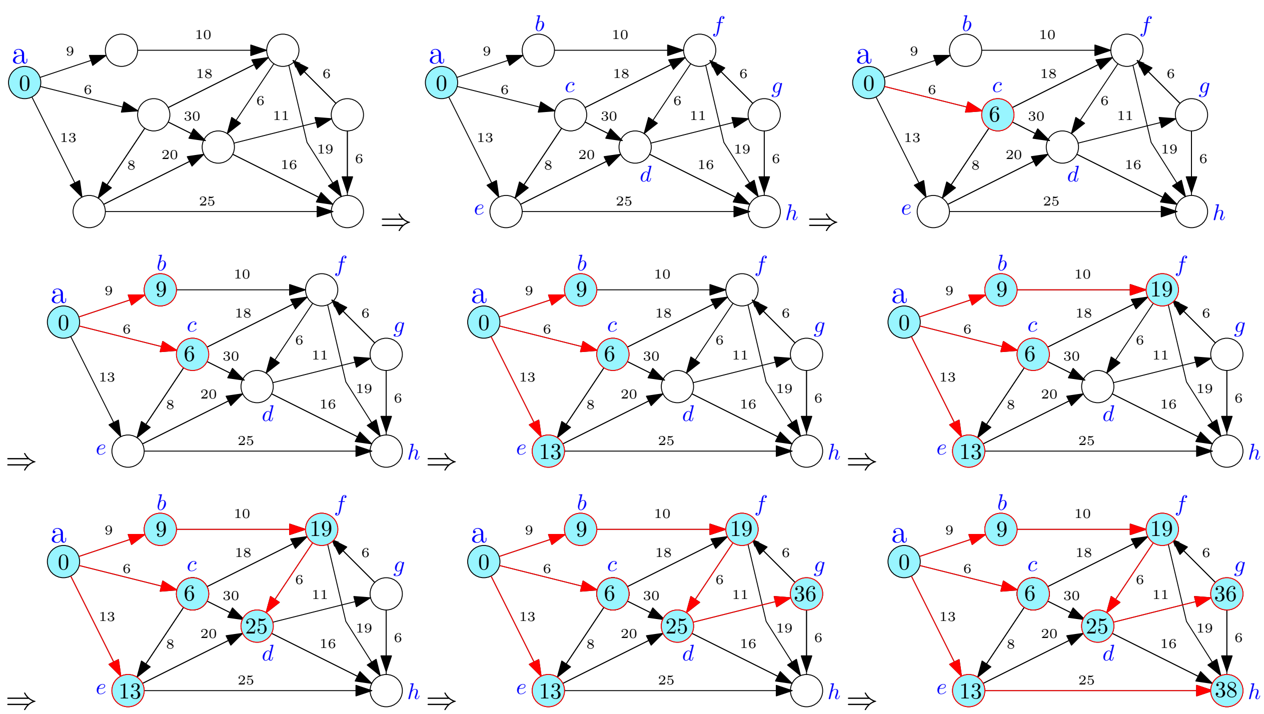
(A) S contains the $i-1$ closest nodes to s
(B) Want to find the ith closest node from $V - S$.
What do we know about the ith closest node?

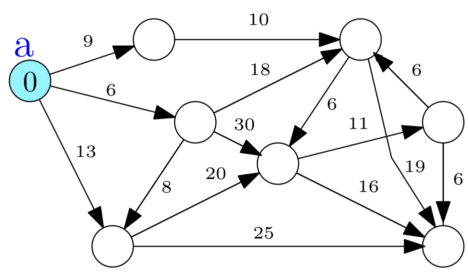
Claim 3.3.2 *Let P be a shortest path from s to v where v is the ith closest node. Then, all intermediate nodes in P belong to S.*

Proof: If P had an intermediate node u not in S then u will be closer to s than v. Implies v is not the ith closest node to s - recall that S already has the $i-1$ closest nodes. ∎

3.3.2 Finding the ith closest node repeatedly

3.3.2.1 An example





3.3.2.2 Finding the ith closest node

Corollary 3.3.3 *The ith closest node is adjacent to S.*

3.3.2.3 Finding the ith closest node

(A) S contains the $i-1$ closest nodes to s
(B) Want to find the ith closest node from $V - S$.
(A) For each $u \in V - S$ let $P(s, u, S)$ be a shortest path from s to u using only nodes in S as intermediate vertices.
(B) Let $d'(s, u)$ be the length of $P(s, u, S)$
Observations: for each $u \in V - S$,
(A) $\text{dist}(s, u) \le d'(s, u)$ since we are constraining the paths
(B) $d'(s, u) = \min_{a \in S}(\text{dist}(s, a) + \ell(a, u))$ - Why?

Lemma 3.3.4 *If v is the ith closest node to s, then $d'(s, v) = \text{dist}(s, v)$.*

3.3.2.4 Finding the ith closest node

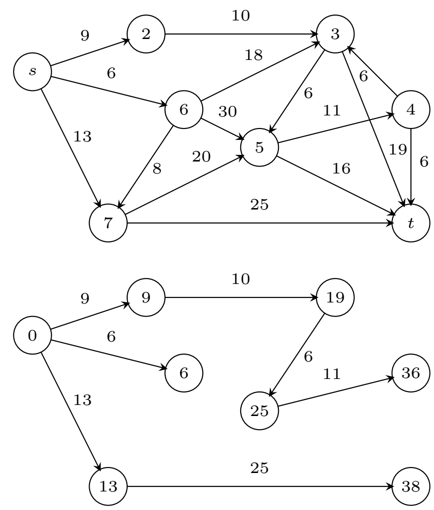
Lemma 3.3.5 *Given:*
(A) S: Set of $i-1$ closest nodes to s.
(B) $d'(s, u) = \min_{x \in S}(\text{dist}(s, x) + \ell(x, u))$
If v is an ith closest node to s, then $d'(s, v) = \text{dist}(s, v)$.

Proof: Let v be the ith closest node to s. Then there is a shortest path P from s to v that contains only nodes in S as intermediate nodes (see previous claim). Therefore $d'(s, v) = \text{dist}(s, v)$. ■

3.3.2.5 Finding the ith closest node

Lemma 3.3.6 *If v is an ith closest node to s, then $d'(s, v) = \text{dist}(s, v)$.*

Corollary 3.3.7 *The ith closest node to s is the node $v \in V - S$ such that $d'(s, v) = \min_{u \in V - S} d'(s, u)$.*



Proof: For every node $u \in V - S$, $\text{dist}(s,u) \le d'(s,u)$ and for the ith closest node v, $\text{dist}(s,v) = d'(s,v)$. Moreover, $\text{dist}(s,u) \ge \text{dist}(s,v)$ for each $u \in V - S$. ■

3.3.2.6 Algorithm

Initialize for each node v: $\text{dist}(s,v) = \infty$
Initialize S = $\emptyset$, $d'(s,s) = 0$
for $i = 1$ to $|V|$ **do**
    (Invariant: S contains the i-1 closest nodes to s *)*
    (Invariant: d'(s,u) is shortest path distance from u to s using only S as intermediate nodes*)*
    Let v be such that $d'(s,v) = \min_{u \in V-S} d'(s,u)$
    $\text{dist}(s,v) = d'(s,v)$
    $S = S \cup \{v\}$
    for each node u in $V \setminus S$ **do**
        $d'(s,u) \Leftarrow \min_{a \in S}\Big(\text{dist}(s,a) + \ell(a,u)\Big)$

Correctness: By induction on i using previous lemmas.
Running time: $O(n \cdot (n+m))$ time.

(A) n outer iterations. In each iteration, $d'(s,u)$ for each u by scanning all edges out of nodes in S; $O(m+n)$ time/iteration.

3.3.2.7 Example

3.3.2.8 Improved Algorithm

(A) Main work is to compute the $d'(s,u)$ values in each iteration
(B) $d'(s,u)$ changes from iteration i to $i+1$ only because of the node v that is added to S in iteration i.

```
Initialize for each node v, dist(s,v) = d'(s,v) = ∞
Initialize S = ∅, d'(s,s) = 0
for i = 1 to |V| do
    // S contains the i − 1 closest nodes to s,
    //       and the values of d'(s,u) are current
    v be node realizing d'(s,v) = min_{u∈V−S} d'(s,u)
    dist(s,v) = d'(s,v)
    S = S ∪ {v}
    Update d'(s,u) for each u in V − S as follows:
        d'(s,u) = min(d'(s,u), dist(s,v) + ℓ(v,u))
```

Running time: $O(m + n^2)$ time.

(A) n outer iterations and in each iteration following steps

(B) updating $d'(s,u)$ after v added takes $O(deg(v))$ time so total work is $O(m)$ since a node enters S only once

(C) Finding v from $d'(s,u)$ values is $O(n)$ time

3.3.2.9 Dijkstra's Algorithm

(A) eliminate $d'(s,u)$ and let $\text{dist}(s,u)$ maintain it

(B) update *dist* values after adding v by scanning edges out of v

```
Initialize for each node v, dist(s,v) = ∞
Initialize S = {s}, dist(s,s) = 0
for i = 1 to |V| do
    Let v be such that dist(s,v) = min_{u∈V−S} dist(s,u)
    S = S ∪ {v}
    for each u in Adj(v) do
        dist(s,u) = min(dist(s,u), dist(s,v) + ℓ(v,u))
```

Priority Queues to maintain *dist* values for faster running time

(A) Using heaps and standard priority queues: $O((m + n)\log n)$

(B) Using Fibonacci heaps: $O(m + n\log n)$.

3.3.3 Priority Queues

3.3.3.1 Priority Queues

Data structure to store a set S of n elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations:

(A) **makePQ**: create an empty queue.

(B) **findMin**: find the minimum key in S.

(C) **extractMin**: Remove $v \in S$ with smallest key and return it.

(D) **insert**$(v, k(v))$: Add new element v with key $k(v)$ to S.

(E) **delete**(v): Remove element v from S.

(F) **decreaseKey**$(v, k'(v))$: *decrease* key of v from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \le k(v)$.

(G) **meld**: merge two separate priority queues into one.

All operations can be performed in $O(\log n)$ time.

decreaseKey is implemented via **delete** and **insert**.

3.3.3.2 Dijkstra's Algorithm using Priority Queues

```
Q ⇐ makePQ()
insert(Q, (s,0))
for each node u ≠ s do
    insert(Q, (u,∞))
S ⇐ ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    S = S ∪ {v}
    for each u in Adj(v) do
        decreaseKey(Q, (u, min(dist(s,u), dist(s,v) + ℓ(v,u)))).
```

Priority Queue operations:

(A) $O(n)$ **insert** operations

(B) $O(n)$ **extractMin** operations

(C) $O(m)$ **decreaseKey** operations

3.3.3.3 Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

(A) All operations can be done in $O(\log n)$ time

Dijkstra's algorithm can be implemented in $O((n + m) \log n)$ time.

3.3.3.4 Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

(A) **extractMin**, **insert**, **delete**, **meld** in $O(\log n)$ time

(B) **decreaseKey** in $O(1)$ *amortized* time: ℓ **decreaseKey** operations for $\ell \geq n$ take *together* $O(\ell)$ time

(C) Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

(A) Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.

(B) Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

3.3.3.5 Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to V.
Question: How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s,0))
prev(s) ⇐ null
for each node u ≠ s do
    insert(Q, (u,∞) )
    prev(u) ⇐ null

S = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    S = S ∪ {v}
    for each u in Adj(v) do
        if (dist(s,v) + ℓ(v,u) < dist(s,u) ) then
            decreaseKey(Q, (u, dist(s,v) + ℓ(v,u)) )
            prev(u) = v
```

3.3.3.6 Shortest Path Tree

Lemma 3.3.8 *The edge set* $(u, \text{prev}(u))$ *is the* reverse *of a shortest path tree rooted at* s*. For each* u*, the reverse of the path from* u *to* s *in the tree is a shortest path from* s *to* u*.*

Proof:[Proof Sketch.]
(A) The edge set $\{(u, \text{prev}(u)) \mid u \in V\}$ induces a directed in-tree rooted at s (Why?)
(B) Use induction on $|S|$ to argue that the tree is a shortest path tree for nodes in V.

∎

3.3.3.7 Shortest paths to s

Dijkstra's algorithm gives shortest paths from s to all nodes in V.
How do we find shortest paths from all of V to s?
(A) In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to distinguish.
(B) In directed graphs, use Dijkstra's algorithm in G^{rev}!