DFS in Directed Graphs, Strong Connected Components, and DAGs

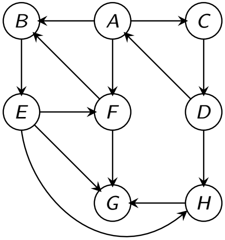
Lecture 2

August 25, 2011

Strong Connected Components (SCCs)

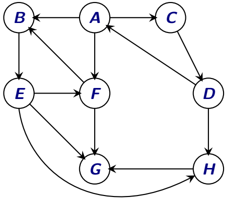
Algorithmic Problem

Find all SCCs of a given directed graph.

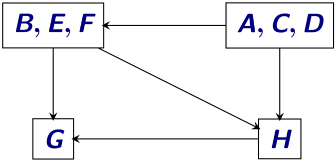


Previous lecture:
Saw an $O(n \cdot (n + m))$ time algorithm.
This lecture: $O(n + m)$ time algorithm.

Graph of SCCs



Graph G



Graph of SCCs G^{SCC}

Meta-graph of SCCs

Let $S_1, S_2, \ldots S_k$ be the strong connected components (i.e., SCCs) of G. The graph of SCCs is G^{SCC}

- Vertices are $S_1, S_2, \ldots S_k$
- There is an edge (S_i, S_j) if there is some $u \in S_i$ and $v \in S_j$ such that (u, v) is an edge in G.

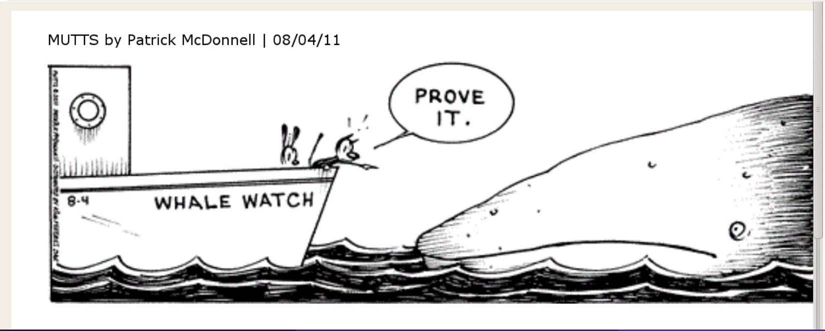
Reversal and SCCs

Proposition

For any graph G, the graph of SCCs of G^{rev} is the same as the reversal of G^{SCC}.

Proof.

Exercise. □



SCCs and DAGs

Proposition

For any graph G, the graph G^{SCC} has no directed cycle.

Proof.

If G^{SCC} has a cycle $S_1, S_2, \ldots, S_k$ then $S_1 \cup S_2 \cup \cdots \cup S_k$ is an SCC in G. Formal details: exercise. $\square$

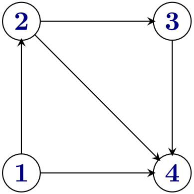
Part I

Directed Acyclic Graphs

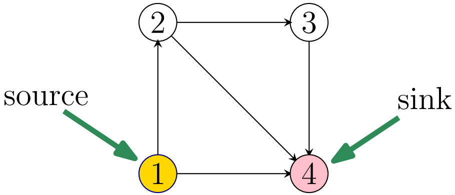
Directed Acyclic Graphs

Definition

A directed graph G is a ***directed acyclic graph*** (DAG) if there is no directed cycle in G.



Sources and Sinks



Definition

- A vertex ***u*** is a ***source*** if it has no in-coming edges.
- A vertex ***u*** is a ***sink*** if it has no out-going edges.

Simple DAG Properties

- Every DAG G has at least one source and at least one sink.
- If G is a DAG if and only if G^{rev} is a DAG.
- G is a DAG if and only each node is in its own strong connected component.

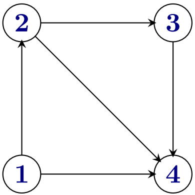
Formal proofs: exercise.

Simple DAG Properties

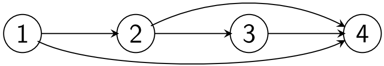
- Every DAG G has at least one source and at least one sink.
- If G is a DAG if and only if $G^{\mathbf{rev}}$ is a DAG.
- G is a DAG if and only each node is in its own strong connected component.

Formal proofs: exercise.

Topological Ordering/Sorting



Graph G



Topological Ordering of G

Definition

A ***topological ordering/topological sorting*** of $G = (V, E)$ is an ordering $<$ on V such that if $(u, v) \in E$ then $u \prec v$.

Informal equivalent definition:

One can order the vertices of the graph along a line (say the x-axis) such that all edges are from left to right.

DAGs and Topological Sort

Lemma

A directed graph G can be topologically ordered iff it is a DAG.

Proof.

$\Longrightarrow$: Suppose G is not a DAG and has a topological ordering $\prec$. G has a cycle $C = u_1, u_2, \ldots, u_k, u_1$.

Then $u_1 \prec u_2 \prec \ldots \prec u_k \prec u_1$!

That is... $u_1 \prec u_1$.

A contradiction (to $\prec$ being an order).

Not possible to topologically order the vertices. $\square$

DAGs and Topological Sort

Lemma

A directed graph G can be topologically ordered iff it is a DAG.

Continued.

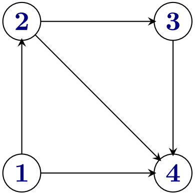
$\Leftarrow$: Consider the following algorithm:

- Pick a source u, output it.
- Remove u and all edges out of u.
- Repeat until graph is empty.
- Exercise: prove this gives an ordering.

$\square$

Exercise: show above algorithm can be implemented in $O(m+n)$ time.

Topological Sort: An Example



Output: 1 2 3 4

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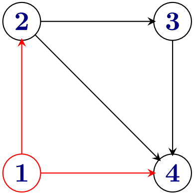

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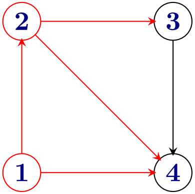

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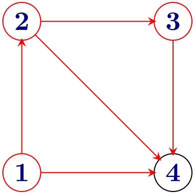

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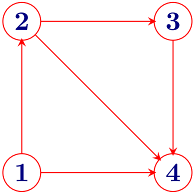

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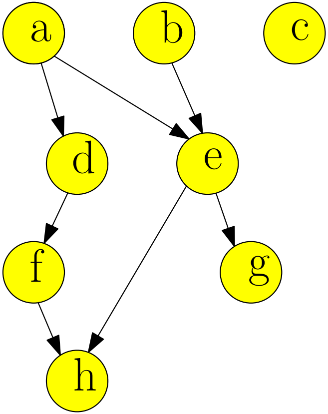
Topological Sort: Another Example



DAGs and Topological Sort

Note: A DAG G may have many different topological sorts.

Question: What is a DAG with the most number of distinct topological sorts for a given number n of vertices?

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Using DFS...

... to check for Acylicity and compute Topological Ordering

Question

Given G, is it a DAG? If it is, generate a topological sort.

DFS based algorithm:

- Compute **DFS**(G)
- If there is a back edge then G is not a DAG.
- Otherwise output nodes in decreasing post-visit order.

Correctness relies on the following:

Proposition

*G is a DAG iff there is no back-edge in **DFS**(G).*

Proposition

If G is a DAG and $\mathrm{post}(v) > \mathrm{post}(u)$, then (u, v) is not in G.

Using DFS...

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Proof

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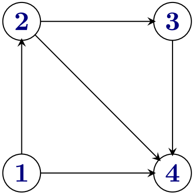
Proof

In lecture notes...

Example



Back edge and Cycles

Proposition

G has a cycle iff there is a back-edge in **DFS**(G).

Proof.

If: (u, v) is a back edge implies there is a cycle C consisting of the path from v to u in **DFS** search tree and the edge (u, v).

Only if: Suppose there is a cycle $C = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \rightarrow v_1$.
Let v_i be first node in C visited in **DFS**.
All other nodes in C are descendants of v_i since they are reachable from v_i.
Therefore, (v_{i-1}, v_i) (or (v_k, v_1) if $i = 1$) is a back edge. □

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DAGs and Partial Orders

Definition

A ***partially ordered set*** is a set $\boldsymbol{S}$ along with a binary relation $\preceq$ such that $\preceq$ is

1. ***reflexive*** ($a \preceq a$ for all $a \in V$),
2. ***anti-symmetric*** ($a \preceq b$ and $a \neq b$ implies $b \not\preceq a$), and
3. ***transitive*** ($a \preceq b$ and $b \preceq c$ implies $a \preceq c$).

Example: For numbers in the plane define $(x, y) \preceq (x', y')$ iff $x \leq x'$ and $y \leq y'$.

Observation: A *finite* partially ordered set is equivalent to a DAG. (No equal elements.)

Observation: A topological sort of a DAG corresponds to a complete (or total) ordering of the underlying partial order.

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What's DAG but a sweet old fashioned notion

Who needs a DAG...

Example

- ***V***: set of ***n*** products (say, ***n*** different types of tablets).
- Want to buy one of them, so you do market research...
- Online reviews compare only pairs of them.
  ...Not everything compared to everything.
- Given this partial information:
  - Decide what is the best product.
  - Decide what is the ordering of products from best to worst.
  - ...

What DAGs got to do with it?

Or why we should care about DAGs

- DAGs enable us to represent partial ordering information we have about some set (very common situation in the real world).
- Questions about DAGs:
  - Is a graph G a DAG?

    $\Longleftrightarrow$

    Is the partial ordering information we have so far is consistent?
  - Compute a topological ordering of a DAG.

    $\Longleftrightarrow$

    Find an a consistent ordering that agrees with our partial information.
  - Find comparisons to do so DAG has a unique topological sort.

    $\Longleftrightarrow$

    Which elements to compare so that we have a consistent ordering of the items.

Part II

Linear time algorithm for finding all strong connected components of a directed graph

Finding all SCCs of a Directed Graph

Problem

Given a directed graph $G = (V, E)$, output *all* its strong connected components.

Straightforward algorithm:

```
Mark all vertices in V as not visited.
for each vertex u ∈ V not visited yet do
    find SCC(G, u) the strong component of u:
        Compute rch(G, u) using DFS(G, u)
        Compute rch(G^rev, u) using DFS(G^rev, u)
        SCC(G, u) ⇐ rch(G, u) ∩ rch(G^rev, u)
        ∀u ∈ SCC(G, u): Mark u as visited.
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Running time: $O(n(n + m))$
Is there an $O(n + m)$ time algorithm?

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```

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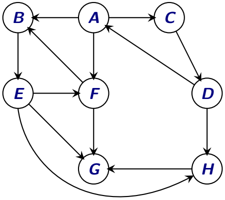
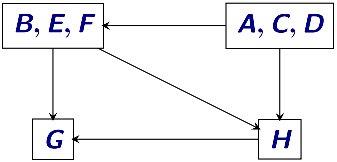
Structure of a Directed Graph



Graph G



Graph of SCCs G^{SCC}

Reminder

G^{SCC} is created by collapsing every strong connected component to a single vertex.

Proposition

For a directed graph G, its meta-graph G^{SCC} is a DAG.

Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph...

Wishful Thinking Algorithm

- Let u be a vertex in a sink SCC of G^{SCC}
- Do **DFS**(u) to compute $\mathrm{SCC}(u)$
- Remove $\mathrm{SCC}(u)$ and repeat

Justification

- **DFS**(u) only visits vertices (and edges) in $\mathrm{SCC}(u)$
- **DFS** done only in G (not in $G^{\mathbf{rev}}$) to compute u strong connected component (SCC). [Magic!]
- **DFS**(u) takes time proportional to size of $\mathrm{SCC}(u)$
- Therefore, total time $O(n+m)$!

Big Challenge(s)

How do we find a vertex in the sink SCC of G^{SCC}?

Can we obtain an *implicit* topological sort of G^{SCC} without computing G^{SCC}?

Answer: **DFS**(G) gives some information!

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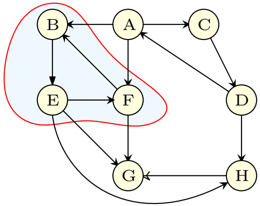
Answer: **DFS**($\boldsymbol{G}$) gives some information!

Post-visit times of SCCs

Definition

Given G and a SCC S of G, define $\mathrm{post}(S) = \max_{u \in S} \mathrm{post}(u)$ where post numbers are with respect to some **DFS**(G).



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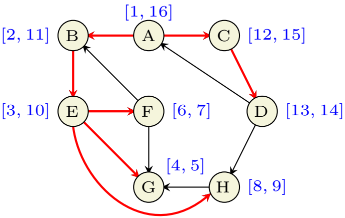

Graph with pre-post times for **DFS**(***A***); black edges in tree

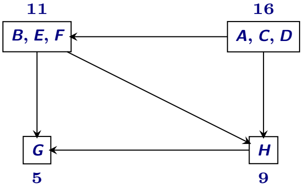

Figure: G^{SCC} with post times

Graph of strong connected components

... and post-visit times

Proposition

If S and S' are SCCs in G and (S, S') is an edge in G^{SCC} then $\mathrm{post}(S) > \mathrm{post}(S')$.

Proof.

Let u be first vertex in $S \cup S'$ that is visited.

- If $u \in S$ then all of S' will be explored before **DFS**(u) completes.
- If $u \in S'$ then all of S' will be explored before any of S.

$\square$

A False Statement: If S and S' are SCCs in G and (S, S') is an edge in G^{SCC} then for *every* $u \in S$ and $u' \in S'$, $\mathrm{post}(u) > \mathrm{post}(u')$.

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Topological ordering of the strong components

Corollary

Ordering SCC*s in decreasing order of* $\mathrm{post}(\boldsymbol{S})$ *gives a topological ordering of* G^{SCC}

Recall: for a DAG, ordering nodes in decreasing post-visit order gives a topological sort.

So...
DFS$(\boldsymbol{G})$ gives some information on topological ordering of $\boldsymbol{G}^{\mathrm{SCC}}$!

Topological ordering of the strong components

Corollary

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Recall: for a DAG, ordering nodes in decreasing post-visit order gives a topological sort.

So...
DFS(G) gives some information on topological ordering of G^{SCC}!

Finding Sources

Proposition

The vertex u with the highest post visit time belongs to a source SCC *in* G^{SCC}

Proof.

- $\mathbf{post}(\mathrm{SCC}(u)) = \mathbf{post}(u)$
- Thus, $\mathbf{post}(\mathrm{SCC}(u))$ is highest and will be output first in topological ordering of G^{SCC}.

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$\square$

Finding Sinks

Proposition

The vertex u with highest post visit time in **DFS**(G^{rev}) *belongs to a sink SCC of G.*

Proof.

- u belongs to source SCC of G^{rev}
- Since graph of SCCs of G^{rev} is the reverse of G^{SCC}, $\mathrm{SCC}(u)$ is sink SCC of G. $\square$

Finding Sinks

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The vertex u with highest post visit time in $\mathbf{DFS}(G^{\mathrm{rev}})$ belongs to a sink SCC of G.

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Linear Time Algorithm

...for computing the strong connected components in G

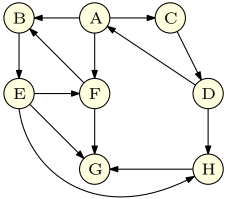
```
do DFS(G^rev) and sort vertices in decreasing post order.
Mark all nodes as unvisited
for each u in the computed order do
    if u is not visited then
        DFS(u)
        Let S_u be the nodes reached by u
        Output S_u as a strong connected component
        Remove S_u from G
```

Analysis

Running time is $O(n+m)$. (Exercise)

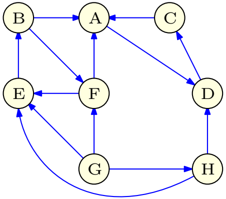
Linear Time Algorithm: An Example - Initial steps

Graph G:



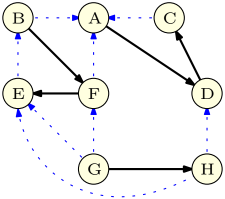
$\Longrightarrow$

Reverse graph $\boldsymbol{G}^{\mathrm{rev}}$:



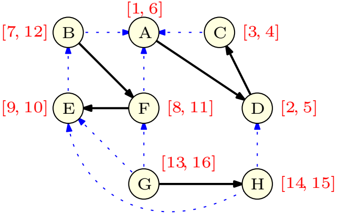
DFS of reverse graph:

$\Longrightarrow$



$\Longrightarrow$

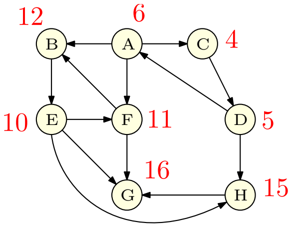
Pre/Post **DFS** numbering of reverse graph:



Linear Time Algorithm: An Example

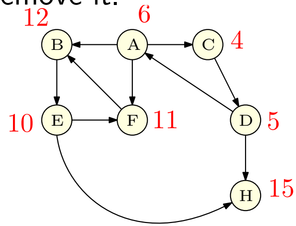
Removing connected components: 1

Original graph G with rev post numbers:



$\Longrightarrow$

Do **DFS** from vertex G remove it.



SCC computed:
$\{\boldsymbol{G}\}$

Linear Time Algorithm: An Example

Removing connected components: 2

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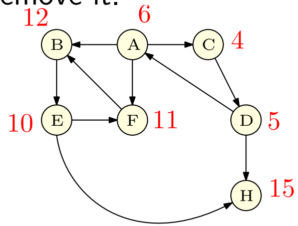
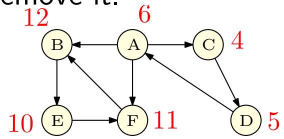


SCC computed:
$\{G\}$

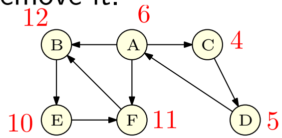
$\Longrightarrow$

Do **DFS** from vertex H, remove it.



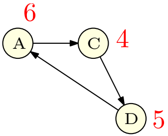
SCC computed:
$\{G\}, \{H\}$

Do **DFS** from vertex H, remove it.



$\Longrightarrow$

Do **DFS** from vertex F
Remove visited vertices: $\{F, B, E\}$.



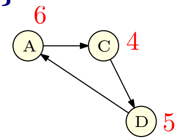
SCC computed:
$\{G\}, \{H\}$

SCC computed:
$\{G\}, \{H\}, \{F, B, E\}$

Linear Time Algorithm: An Example

Removing connected components: 4

Do **DFS** from vertex F
Remove visited vertices:
$\{F, B, E\}$.

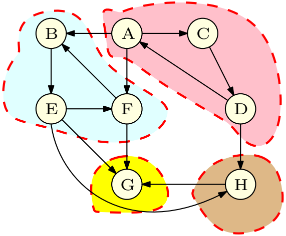


$\Longrightarrow$

Do **DFS** from vertex A
Remove visited vertices:
$\{A, C, D\}$.

SCC computed:
$\{G\}, \{H\}, \{F, B, E\}$

SCC computed:
$\{G\}, \{H\}, \{F, B, E\}, \{A, C, D\}$



SCC computed:
$\{G\}, \{H\}, \{F, B, E\}, \{A, C, D\}$
Which is the correct answer!

Obtaining the meta-graph...

Once the strong connected components are computed.

Exercise:

Given all the strong connected components of a directed graph $G = (V, E)$ show that the meta-graph G^{SCC} can be obtained in $O(m + n)$ time.

Correctness: more details

- let $S_1, S_2, \ldots, S_k$ be strong components in G
- Strong components of G^{rev} and G are same and meta-graph of G is reverse of meta-graph of G^{rev}.
- consider **DFS**(G^{rev}) and let $u_1, u_2, \ldots, u_k$ be such that $\mathrm{post}(u_i) = \mathrm{post}(S_i) = \max_{v \in S_i} \mathrm{post}(v)$.
- Assume without loss of generality that $post(u_k) > post(u_{k-1}) \geq \ldots \geq post(u_1)$ (renumber otherwise). Then $S_k, S_{k-1}, \ldots, S_1$ is a topological sort of meta-graph of G^{rev} and hence $S_1, S_2, \ldots, S_k$ is a topological sort of the meta-graph of G.
- u_k has highest post number and **DFS**(u_k) will explore all of S_k which is a sink component in G.
- After S_k is removed u_{k-1} has highest post number and **DFS**(u_{k-1}) will explore all of S_{k-1} which is a sink component in remaining graph $G - S_k$. Formal proof by induction.

Part III

An Application to `make`

`make` Utility [Feldman]

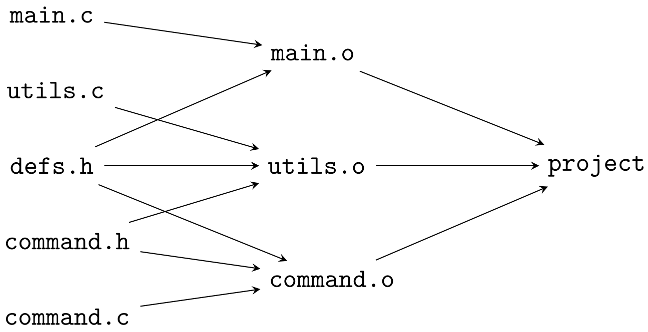
- Unix utility for automatically building large software applications
- A makefile specifies
  - Object files to be created,
  - Source/object files to be used in creation, and
  - How to create them

An Example `makefile`

```
project: main.o utils.o command.o
	cc -o project main.o utils.o command.o

main.o: main.c defs.h
	cc -c main.c
utils.o: utils.c defs.h command.h
	cc -c utils.c
command.o: command.c defs.h command.h
	cc -c command.c
```

`makefile` as a Digraph



Computational Problems for `make`

- Is the `makefile` reasonable?
- If it is reasonable, in what order should the object files be created?
- If it is not reasonable, provide helpful debugging information.
- If some file is modified, find the fewest compilations needed to make application consistent.

Algorithms for `make`

- Is the `makefile` reasonable? Is G a DAG?
- If it is reasonable, in what order should the object files be created? Find a topological sort of a DAG.
- If it is not reasonable, provide helpful debugging information. Output a cycle. More generally, output all strong connected components.
- If some file is modified, find the fewest compilations needed to make application consistent.
  - Find all vertices reachable (using **DFS**/**BFS**) from modified files in directed graph, and recompile them in proper order. Verify that one can find the files to recompile and the ordering in linear time.

Take away Points

- Given a directed graph G, its SCCs and the associated acyclic meta-graph G^{SCC} give a structural decomposition of G that should be kept in mind.
- There is a **DFS** based linear time algorithm to compute all the SCCs and the meta-graph. Properties of **DFS** crucial for the algorithm.
- DAGs arise in many application and topological sort is a key property in algorithm design. Linear time algorithms to compute a topological sort (there can be many possible orderings so not unique).

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