CS 473: Fundamental Algorithms, Fall 2011

Homework 9 (due Monday, 23:55:00, November 14, 2011)

Collaboration Policy & submission guidelines: See homework 1.
Each student individually have to also do **quiz 9** online.

Version: **1.1**

1. (30 PTS.) Independence Matrix

   Consider a $0-1$ matrix H with n_1 rows and n_2 columns. We refer to a row or a column of the matrix H as a line. We say that a set of 1's in the matrix H is *independent* if no two of them appear in the same line. We also say that a set of lines in the matrix is a *cover* of H if they include (i.e., "cover") all the 1's in the matrix. Using the max-flow min-cut theorem on an appropriately defined network, show that the maximum number of independent 1's equals the minimum number of lines in the cover.

2. (40 PTS.) Min cost flow.

   In the following, you can assume all the numbers under considerations are integers.

   Consider a flow network with **finite** capacities on the edges. You have to send k units of flow (k is an integer) from s to t in this network, and the twist is that there are costs associated with sending one unit of flow on edge. That is, for every edge e of G, there is a cost $\mathrm{cost}(e) \geq 0$ associated with it (which is a positive integer number). Formally, given a flow $f(\cdot)$ defined on the edges, the ***cost*** of this flow is $\sum_{e \in E(G)} f(e)\mathrm{cost}(e)$.

   (A) (5 PTS.) Describe an algorithm that computes a flow of k units from s to t in the given network.
   (B) (5 PTS.) Describe how to compute the residual network of such a flow. What are the costs on the forward and backward edges?
   (C) (10 PTS.) Given the residual network, show how to compute a negative cycle (i.e., a cycle such that the total cost of the edges along the cycle is negative) in this graph if such a cycle exists. What is the running time of your algorithm? (Any polynomial time algorithm here would be good enough.)
   (D) (5 PTS.) Argue, that if one can find such a negative cycle, then one can compute a cheaper flow than the current flow.
   (E) (5 PTS.) Let f be a flow sending k units of flow from s to t, which is not a minimum cost flow. Prove, that there must be a negative cycle in the residual graph of f. (Hint: Prove it first for the case $k = 1$.)
   (F) (10 PTS.) Let W be the maximum cost of an edge in the given network G. Describe an algorithm (as fast as possible [you would lose 2 points if your algorithm is polynomial in n, k and W but slower than our solution]) that compute the min-cost flow in G from s to t that carries k units of flow.

3. (30 PTS.) Flow my tears the policeman said.

   For this question, you would need your solution from question 2.

   (a) (10 PTS.) Given a directed graph with positive integer costs on the edges (say, the maximum cost is W) and two vertices s and t in the graph, describe how to compute the

k edge disjoint paths from s to t, such that the total cost of these paths is minimized. What is the running time of your algorithm?

(b) (10 PTS.) You are given a bipartite graph G with n vertices and m edges. Every edge has a cost which is a positive integer number. Describe an algorithm that decides if this graph has a perfect matching, and if so, outputs the cheapest such perfect matching. What is the running time of your algorithm as a function of n, m, and W (here, W is the maximum weight of an edge in the graph).

(c) (10 PTS.) Banana just released a new version of their iFifi – the first electronic gizmo that not only can surf the web, but it is also dishwasher safe (not to mention that it comes in two colors: black and blacker). Banana has k distribution centers $C_1, \ldots, C_k$ in the US, and you know for each one of them how many iFifi they currently have in stock (i.e., $t_1, \ldots, t_k$). You need to plan the distribution of the iFifis to the Banana stores. You have a list of n stores $S_1, \ldots, S_n$, and for each one of them there is a quota f_i of how many iFifis they need. For every distribution center C_i and a store S_j, you know the distance d_{ij} between them in miles (rounded up so it is an integer).

Sending a single iFifi from a distribution center C_i to a store S_i costs $10^{-4}d_{ij}$ dollars. Describe an algorithm, as efficient as possible, that computes the minimum cost way to send all the required iFifis from the distribution centers to the stores. How fast is your algorithm (for this question you can assume the US diameter is 3000 miles).