

CS 473: Algorithms

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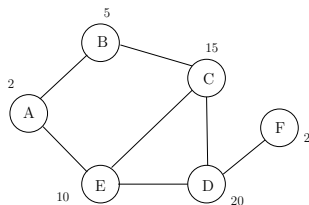
Part I

Maximum Weighted Independent Set in Trees

Maximum Weight Independent Set Problem

Input Graph $G = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

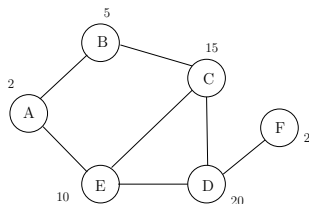
Goal Find maximum weight independent set in G



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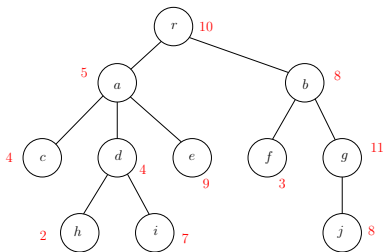


Maximum weight independent set in above graph: $\{B, D\}$

Maximum Weight Independent Set in a Tree

Input Tree $T = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal Find maximum weight independent set in T



Maximum weight independent set in above tree: ??

Towards a Recursive Solution

For an arbitrary graph G :

- Number vertices as v_1, v_2, \dots, v_n
- Find recursively optimum solutions without v_n (recurse on $G - v_n$) and with v_n (recurse on $G - v_n - N(v_n)$ & include v_n).
- Saw that if graph G is arbitrary there was no good ordering that resulted in a small number of subproblems.

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What about a tree? Natural candidate for v_n is root r of T ?

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Natural candidate for v_n is root r of T ? Let \mathcal{O} be an optimum solution to the whole problem.

Case $r \notin \mathcal{O}$ Then \mathcal{O} contains an optimum solution for each subtree of T hanging at a child of r .

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- Case $r \in \mathcal{O}$ None of the children of r can be in \mathcal{O} . $\mathcal{O} - \{r\}$ contains an optimum solution for each subtree of T hanging at a grandchild of r .

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Subproblems? Subtrees of T hanging at nodes in T .

A Recursive Solution

$T(u)$: subtree of T hanging at node u

$OPT(u)$: max weighted independent set value in $T(u)$

$OPT(u) =$

A Recursive Solution

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$OPT(u)$: max weighted independent set value in $T(u)$

$$OPT(u) = \max\left\{ \sum_{v \text{ child of } u} OPT(v), w(u) + \sum_{v \text{ grandchild of } u} OPT(v) \right\}$$

Iterative Algorithm

- Compute $OPT(u)$ bottom up. To evaluate $OPT(u)$ need to have computed values of all children and grandchildren of u
- What is an ordering of nodes of a tree T to achieve above?

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Post-order traversal of a tree.

Iterative Algorithm

Iterative Algorithm

MIS-Tree(T):

Let v_1, v_2, \dots, v_n be a post-order traversal of nodes of T

for $i = 1$ to n do

$$M[v_i] = \max\left(\sum_{v_j \text{ child of } v_i} M[v_j], w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j]\right)$$

end for

return $M[v_n]$ (* Note: v_n is the root of T *)

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Space: $O(n)$ to store the value at each node of T

Running time:

Iterative Algorithm

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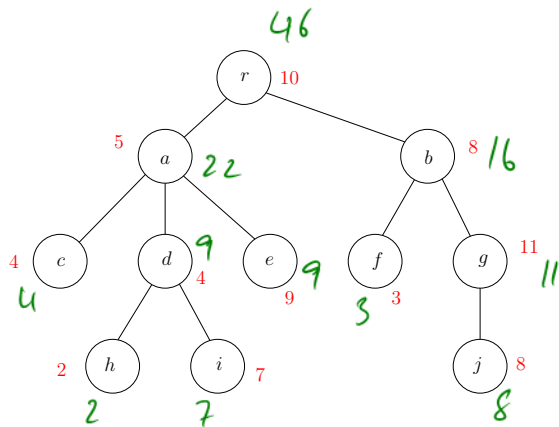
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Running time:

- Naive bound: $O(n^2)$ since each $M[v_i]$ evaluation may take $O(n)$ time and there are n evaluations.
- Better bound: $O(n)$. A value $M[v_j]$ is accessed only by its parent and grand parent.

Example



Part II

DAGs and Dynamic Programming

Observation

Let A be a recursive algorithm for problem Π . For each instance I of Π there is an associated DAG $G(I)$.

- Create directed graph $G(I)$ as follows
- For each sub-problem in the execution of A on I create a node
- If sub-problem v *depends on* or *recursively calls* sub-problem u add *directed* edge (u, v) to graph
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- $G(I)$ is a DAG. Why? If $G(I)$ has a cycle then A will not terminate on I

Observation

An iterative algorithm B obtained from a recursive algorithm A for a problem Π does the following: for each instance I of Π , it computes a topological sort of $G(I)$ and evaluates sub-problems according to the topological ordering.

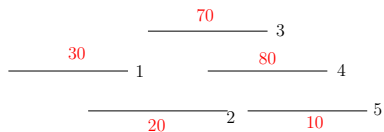
- Sometimes the DAG $G(I)$ can be obtained directly without thinking about the recursive algorithm A
- In some cases (**not all**) the computation of an optimal solution reduces to a shortest/longest path in DAG $G(I)$
- Topological sort based shortest/longest path computation is dynamic programming!

Weighted Interval Scheduling via Longest Path in a DAG

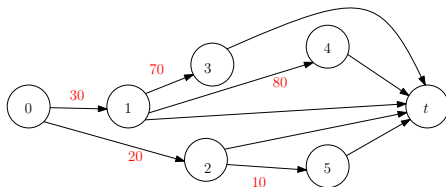
Given intervals, create a DAG as follows

- one node for each interval plus a dummy source node for interval 0 plus a dummy sink node t .
- for each interval i add edge $(p(i), i)$ of length/weight v_i
- for each interval i add edge (i, t) of length 0

Example



$$p(5) = 2, p(4) = 1, p(3) = 1, p(2) = 0, p(1) = 0$$



Relating Optimum Solution

Given interval problem instance I let $G(I)$ denote the DAG constructed as described.

Claim: Optimum solution to weighted interval scheduling instance I is given by *longest* path from s to t in $G(I)$.

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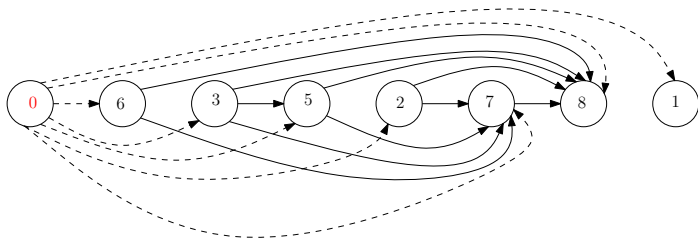
Assuming claim is true,

- If I has n intervals, DAG $G(I)$ has $n + 2$ nodes and $O(n)$ edges. Creating $G(I)$ takes $O(n \log n)$ time: to find $p(i)$ for each i . How?
- Longest path can be computed in $O(n)$ time — recall $O(m + n)$ algorithm for shortest/longest paths in DAGs.

DAG for Longest Increasing Sequence

Given sequence a_1, a_2, \dots, a_n create DAG as follows:

- add sentinel a_0 to sequence where a_0 is less than smallest element in sequence
- for each i there is a node v_i
- if $i < j$ and $a_i < a_j$ add an edge (v_i, v_j)
- find longest path from v_0



Part III

Edit Distance and Sequence Alignment

Spell Checking Problem

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Question: Given two strings $x_1x_2 \dots x_n$ and $y_1y_2 \dots y_m$ what is a *distance* between them?

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Edit Distance: minimum number of “edits” to transform x into y .

Definition

Edit distance between two words X and Y is the number of letter insertions, letter deletions and letter substitutions required to obtain Y from X .

Example

The edit distance between FOOD and MONEY is at most 4

FOOD \rightarrow MOOD \rightarrow MON \square D \rightarrow MONED \rightarrow MONEY

Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

F	O	O		D
M	O	N	E	Y

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Formally, an **alignment** is a set M of pairs (i, j) such that each index appears at most once, and there is no “crossing”: $i < i'$ and i is matched to j implies i' is matched to $j' > j$. In the above example, this is $M = \{(1, 1), (2, 2), (3, 3), (4, 5)\}$.

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Edit Distance Problem

Problem

Given two words, find the edit distance between them, i.e., an alignment of smallest cost.

- Spell-checkers and Dictionaries

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- DNA sequence alignment

- Spell-checkers and Dictionaries
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- DNA sequence alignment ... but, we need a new metric

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Edit distance is special case when $\delta = \alpha_{pq} = 1$

An Example

Example

o c u r r a n c e

o c c u r r e n c e

$$\text{Cost} = \delta + \alpha_{ae}$$

o c u r r a n c e

o c c u r r e n c e

$$\text{Cost} = 3\delta$$

Sequence Alignment

Input Given two words X and Y , and gap penalty δ and mismatch costs α_{pq}

Goal Find alignment of minimum cost

Observation

Let $X = x_1x_2 \cdots x_m$ and $Y = y_1y_2 \cdots y_n$. If (m, n) are not matched then either the m 'th position of X remains unmatched or the n 'th position of Y remains unmatched.

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$x_1 \cdots x_m$
 $y_1 \cdots y_n \downarrow$

$x_1 \cdots x_m \uparrow$
 $y_1 \cdots y_{n-1}$

Optimal Costs

Let $\text{Opt}(i, j)$ be optimal cost of aligning $x_1 \cdots x_i$ and $y_1 \cdots y_j$.
Then

$$\text{Opt}(i, j) = \min(\alpha_{x_i y_j} + \text{Opt}(i-1, j-1), \delta + \text{Opt}(i-1, j), \delta + \text{Opt}(i, j-1))$$

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Base Cases: $\text{Opt}(i, 0) = \delta \cdot i$ and $\text{Opt}(0, j) = \delta \cdot j$

Dynamic Programming Solution

```
for all i M[i,0] = iδ
for all j M[0,j] = jδ
for i = 1 to m
  for j = 1 to n
    M[i,j] = min (αxiyj + M[i-1,j-1], δ + M[i-1,j], δ + M[i,j-1])
```

Analysis

Dynamic Programming Solution

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- Running time is $O(mn)$

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Analysis

- Running time is $O(mn)$
- Space used is $O(mn)$

Matrix and DAG of Computation

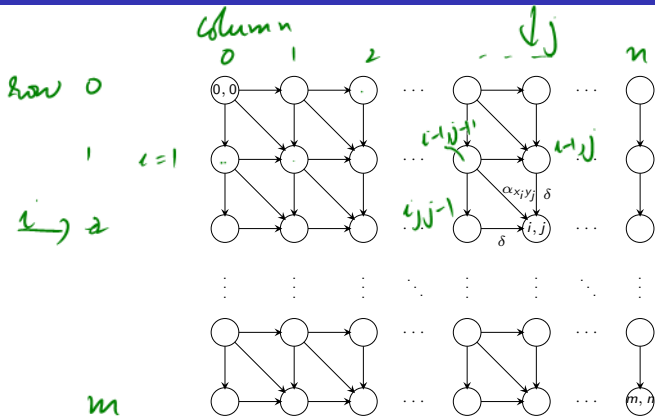


Figure: Iterative algorithm in previous slide computes values in row order. Optimal value is a shortest path from (0,0) to (m,n) in DAG.

Sequence Alignment in Practice

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- So about 10^{10} ops and 10^{10} bytes needed
- The killer is the 10GB storage
- Can we reduce space requirements?

- Recall

$$M(i, j) = \min(\alpha_{x_i y_j} + M(i-1, j-1), \delta + M(i-1, j), \delta + M(i, j-1))$$

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- Entries in j th column only depend on $(j-1)$ 'st column and earlier entries in j th column
- Only store the current column and the previous column reusing space; $N(i, 0)$ stores $M(i, j-1)$ and $N(i, 1)$ stores $M(i, j)$

Computing in column order to save space

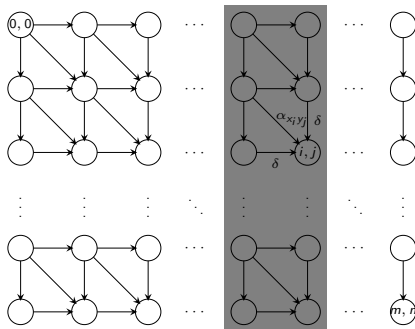


Figure: $M(i, j)$ only depends on previous column values. Keep only two columns and compute in column order.

Space Efficient Algorithm

```
for all i N[i,0] = iδ
for j = 1 to n
  N[0,1] = jδ (* corresponds to M(0,j) *)
  for i = 1 to m
    N[i,1] = min (αxiyj + N[i-1,0], δ + N[i-1,1], δ + N[i,0])
  update N[i,0] = N[i,1]
```

Analysis

Running time is $O(mn)$ and space used is $O(2m) = O(m)$

Analyzing Space Efficiency

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- From the $m \times n$ matrix M we can construct the actual alignment (exercise)
- Matrix N computes cost of optimal alignment but no way to construct the actual alignment
- Space efficient computation of alignment? More complicated algorithm — see text book.

Takeaway Points

- Dynamic programming is based on finding a recursive way to solve the problem. Need a recursion that generates a small number of subproblems.
- Given a recursive algorithm there is a natural DAG associated with the subproblems that are generated for given instance; this is the dependency graph. An iterative algorithm simply evaluates the subproblems in some topological sort of this DAG.
- The space required to evaluate the answer can be reduced in some cases by a careful examination of that dependency DAG of the subproblems and keeping only a subset of the DAG at any time.