CS 473: Algorithms

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Part I

Longest Increasing Subsequence

Sequences

Definition

Sequence: an ordered list $a_1, a_2, \ldots, a_n$. *Length* of a sequence is number of elements in the list.

Definition

$a_{i_1}, \ldots, a_{i_k}$ is a subsequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < \ldots < i_k \leq n$.

Definition

A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Subsequence: 5, 2, 1

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an *increasing subsequence* $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

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Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

Näive Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
LIS(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max= |B|
    end for
    Output max
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Running time:

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```

Running time: $O(n2^n)$.
2^n subsequences of a sequence of lengh n and $O(n)$ time to check if a given sequence is increasing.

Recursive Approach: Take 1

Can we find a recursive algorithm for LIS?

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Case 1 : does not contain a_n in which case $\text{LIS}(A[1..n]) = \text{LIS}(A[1..(n-1)])$

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Observation: if a_n is in the longest increasing subsequence then all the elements before it must be smaller.

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    m = LIS(A[1..(n-1))
    B is subsequence of A[1..(n-1)] with only elements less than a_n
    (* let h be size of B, h ≤ n-1 *)
    m = max(m, 1+LIS(B[1..h]))
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Recursion for running time: $T(n) \leq 2T(n-1) + O(n)$.
$T(n) = O(2^n)$. Easy to see that it is $O(n2^n)$.

Recursive Approach: Take 2

$\text{LIS}(A[1..n])$:

Case 1 : does not contain a_n in which case
$\text{LIS}(A[1..n]) = \text{LIS}(A[1..(n-1)])$

Case 2 : contains a_n in which case $\text{LIS}(A[1..n])$ is not so clear.

Observation: For second case we want to find a subsequence in $A[1..(n-1)]$ that is restriced to numbers less than a_n. This suggests that a more general problem is $\text{LIS-smaller}(A[1..n], x)$ which gives the longest increasing subsequence in A where each number in the sequence is less than x.

Recursive Approach: Take 2

LIS-smaller($A[1..n], x$) : length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

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LIS-smaller(A[1..n],x):
    if (n = 0) return 0
    m = LIS-smaller(A[1..(n-1)],x)
    if (A[n] < x)
        m = max(m, 1+LIS-smaller(A[1..(n-1)],A[n]))
    Output m

LIS(A[1..n]):
    return LIS-smaller(A[1..n],∞)
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Recursion for running time: $T(n) \leq 2T(n-1) + O(n)$.

Question: Is there any advantage?

Recursive Algorithm: Take 2

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Question: What are the recursive subproblem generated by LIS-smaller$(A[1..n], x)$?

- For $0 \leq i < n$ LIS-smaller$(A[1..i], y)$ where y is either x or one of $A[i+1], \ldots, A[n]$.

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- For $0 \leq i < n$ LIS-smaller($A[1..i], y$) where y is either x or one of $A[i+1], \ldots, A[n]$.

Observation: previous recursion also generates only $O(n^2)$ subproblems. Slightly harder to see.

Recursive Algorithm: Take 3

LIS-Ending($A[1..n]$): length of longest increasing sub-sequence that *ends* in $A[n]$.

Question: can we obtain a recursive expression?

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Question: can we obtain a recursive expression?

$$\text{LIS-Ending}(A[1..n]) = \max_{i:A[i]<A[n]} (1 + \text{LIS-Ending}(A[1..i]))$$

Recursive Algorithm: Take 3

```
LIS-Ending(A[1..n]):
    if (n = 0) return 0
    m = 1
    for i = 1 to n - 1 do
        if (A[i] < A[n]) do
            m = max(m, 1+LIS-Ending(A[1..i]))

    Output m
```

`LIS(A[1..n]):`

  return $max_{i=1}^{n}$ `LIS-Ending(A[1..`~~n~~ i`])`

Recursive Algorithm: Take 3

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    Output m

LIS(A[1..n]):
    return max_{i=1}^{n} LIS-Ending(A[1..n],∞)
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Question: How many distinct subproblems generated by LIS-Ending($A[1..n]$)?

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Question: How many distinct subproblems generated by LIS-Ending($A[1..n]$)? n.

Iterative Algorithm via Memoization

Compute the values LIS-Ending($A[1..i]$) iteratively in a bottom up fashion.

```
LIS-Ending(A[1..n]):
    Array L[1..n] (* L[i] stores the value LIS-Ending(A[1..i]) *)
    for i = 1 to n do
        L[i] = 1
        for j = 1 to i-1 do
            if (A[j] < A[i]) do
                L[i] = max (L[i], 1+ L[j])
    return L

LIS(A[1..n]):
    L = LIS-Ending(A[1..n])
    return the maximum value in L
```

Iterative Algorithm via Memoization

Simplifying:

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    Output m
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Correctness: Via induction following the recursion

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Running time: $O(n^2)$,

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Correctness: Via induction following the recursion
Running time: $O(n^2)$, Space:

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```

Correctness: Via induction following the recursion

Running time: $O(n^2)$, Space: $\Theta(n)$

Example

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- Sequence: 6, 3, 5, 2, 7, 8, ~~1~~
- Longest increasing subsequence: 3, 5, 7, 8

6

$L[1] = 1$
$L[2] = 1$
$L[3] = 2$
$L[4] = 1$
$L[5] = 3$
$L[6] = 4$

$L[7] = 1$

4

Example

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Longest increasing subsequence: 3, 5, 7, 8

- $L[i]$ is value of longest increasing subsequence ending in $A[i]$
- Recursive algorithm computes $L[i]$ from $L[1]$ to $L[i-1]$
- Iterative algorithm builds up the values from $L[1]$ to $L[n]$

Memoizing the Second Recursive Algorithm

```
LIS(A[1..n]):
    A[n+1] = ∞ (* add a sentinel at the end *)
    Array L[(n+1),(n+1)] (* two-dimensional array*)
(* L[i,j] for j ≥ i stores the value LIS-Smaller(A[1..i],A[j]) *)
    for j = 1 to n+1 do
        L[0,j] = 0
    for i = 1 to n+1 do
        for j = i to n+1 do
            L[i,j] = L[i-1,j]
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    Output L[n,(n+1)]
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Running time: $O(n^2)$,

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    Output L[n,(n+1)]
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Correctness: Via induction following the recursion (take 2)
Running time: $O(n^2)$, Space: $\Theta(n^2)$

Dynamic Programming

- Find a "smart" recursion for the problem in which the number of distinct subproblems is small; polynomial in the original problem size.
- Eliminate recursion and find an iterative algorithm to compute the problems bottom up by storing the intermediate values in an appropriate data structure; need to find the right way or order the subproblem evaluation.
- Estimate the number of subproblems, the time to evaluate each subproblem and the space needed to store the value. Evaluate the total running time.
- Optimize the resulting algorithm further

Part II

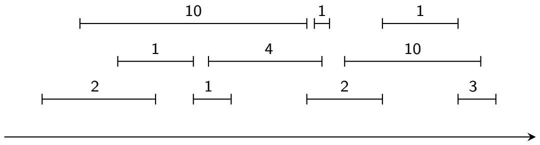
Weighted Interval Scheduling

Weighted Interval Scheduling

Input A set of jobs with start times, finish times and *weights* (or profits)

Goal Schedule jobs so that total weight of jobs is maximized

- Two jobs with overlapping intervals cannot both be scheduled!



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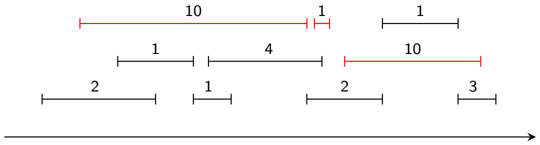
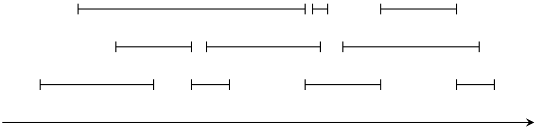


Interval Scheduling

Input A set of jobs with start and finish times to be scheduled on a resource; special case where all jobs have weight 1

Goal Schedule as many jobs as possible



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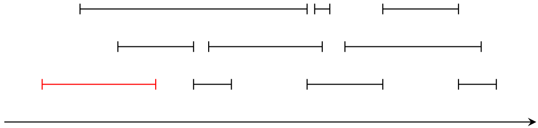

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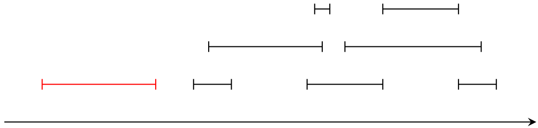

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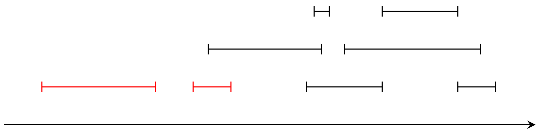

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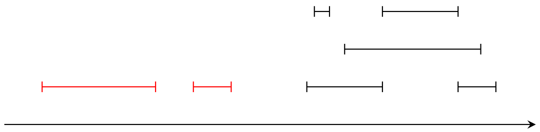

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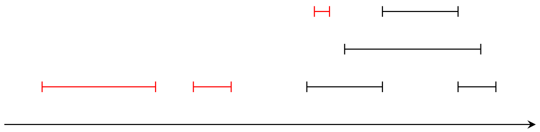

Interval Scheduling

Input A set of jobs with start and finish times to be scheduled on a resource; special case where all jobs have weight 1

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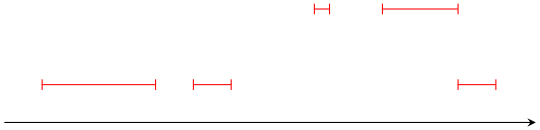

Greedy Strategies

- Earliest finish time first
- Largest weight/profit first
- Largest weight to length ratio first
- Shortest length first
- ...

None of the above strateies lead to an optimum solution.

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Moral: Greedy strategies often don't work!

Reduction to Max Weight Independent Set Problem

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- Given weighted interval scheduling instance I create an instance of max weight independent set on a graph $G(I)$ as follows.
  - For each interval i create a vertex v_i with weight w_i.
  - Add an edge between v_i and v_j if i and j overlap.
- Claim: max weight independent set in $G(I)$ has weight equal to max weight set of intervals in I that do not overlap

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We do not know an efficient (polynomial time) algorithm for independent set! Can we take advantage of the interval structure to find an efficient algorithm?

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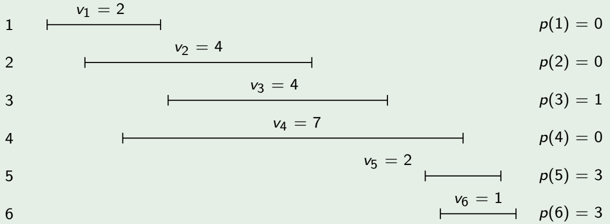
Example



Towards a Recursive Solution

Observation

Consider an optimal schedule $\mathcal{O}$

Case $n \in \mathcal{O}$ *None of the jobs between n and $p(n)$ can be scheduled. Moreover $\mathcal{O}$ must contain an optimal schedule for the first $p(n)$ jobs.*

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Case $n \in \mathcal{O}$ *None of the jobs between n and $p(n)$ can be scheduled. Moreover $\mathcal{O}$ must contain an optimal schedule for the first $p(n)$ jobs.*

Case $n \notin \mathcal{O}$ *$\mathcal{O}$ is an optimal schedule for the first $n - 1$ jobs.*

A Recursive Algorithm

Let O_i be value of an optimal schedule for the first i jobs.

```
Recursively compute O_{p(n)}
Recursively compute O_{n-1}
If (O_{p(n)} + v_n < O_{n-1}) then
    O_n = O_{n-1}
else
    O_n = O_{p(n)} + v_n
Output O_n
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Time Analysis

Running time is $T(n) = T(p(n)) + T(n-1) + O(1)$ which is ...

Bad Example

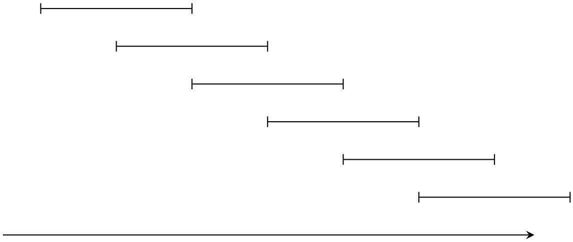

Figure: Bad instance for recursive algorithm

Running time on this instance is

$$T(n) = T(n-1) + T(n-2) + O(1)$$

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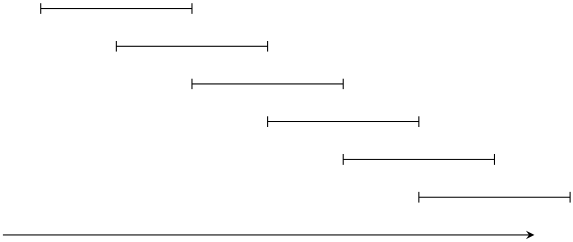

Figure: Bad instance for recursive algorithm

Running time on this instance is

$$T(n) = T(n-1) + T(n-2) + O(1) = \Theta(\phi^n)$$

where $\phi \approx 1.618$ is the golden ratio.

Analysis of the Problem

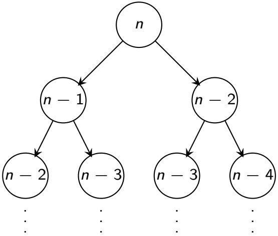

Figure: Label of node indicates size of sub-problem. Tree of sub-problems grows very quickly

Memo(r)ization

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Observation

- *Number of different sub-problems in recursive algorithm is $O(n)$; they are $O_1, O_2, \ldots, O_{n-1}$*
- *Exponential time is due to recomputation of solutions to sub-problems*

Solution

Store optimal solution to different sub-problems, and perform recursive call only if not already computed.

Recursive Solution with Memoization

```
computeOpt(int j)
    if j = 0 then return 0
    if M[j] is defined then (* sub-problem already solved *)
        return M[j]
    if M[j] is not defined then
        M[j] = max(v_j + computeOpt(p(j)), computeOpt(j-1))
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- Each invocation, $O(1)$ time plus: either return a computed value, or generate 2 recursive calls and fill one M[$\cdot$]

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- Each invocation, $O(1)$ time plus: either return a computed value, or generate 2 recursive calls and fill one `M[·]`
- Initially no entry of `M[]` is filled; at the end all entries of `M[]` are filled
- So total time is $O(n)$

Automatic Memoization

Fact

Many functional languages (like LISP) automatically do memoization for recursive function calls!

Back to Weighted Interval Scheduling

Iterative Solution

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M[0] = 0
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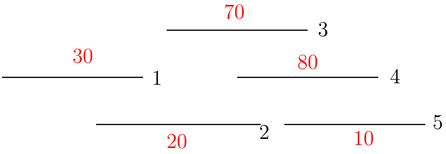
M: table of subproblems

- Implicitly dynamic programming fills the values of M
- Recursion determines order in which table is filled up
- Think of decomposing problem first (recursion) and then worry about setting up table — this comes naturally from recursion

Example



$$p(5) = 2,\ p(4) = 1,\ p(3) = 1,\ p(2) = 0,\ p(1) = 0$$

Computing Solutions

- Memoization + Recursion/Iteration allows one to compute the optimal value. What about the actual schedule?

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Computing Solutions: First Attempt

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```

- Naïvely updating S[] takes $O(n)$ time
- Total running time is $O(n^2)$

Computing Implicit Solutions

Observation

Solution can be obtained from `M[]` *in* $O(n)$ *time, without any additional information*

```
findSolution(int j)
    if (j=0) then return empty schedule
    if (v_j + M[p(j)] > M[j-1]) then
        return findSolution(p(j)) ∪ {j}
    else
        return findSolution(j-1)
```

Makes $O(n)$ *recursive calls, so* `findSolution` *runs in* $O(n)$ *time.*

Computing Implicit Solutions

A generic strategy for computing solutions in dynamic programming:

- keep track of the *decision* in computing the optimum value of a sub-problem. decision space depends on recursion
- once the optimum values are computed, go back and use the decision values to compute an optimum solution.

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Question: What is the decision in computing $M[i]$? Whether to include i or not.

Computing Implicit Solutions

```
M[0] = 0
for i = 1 to n
    M[i] = max(v_i + M[p(i)], M[i-1])
    if (v_i + M[p(i)] > M[i-1])
        Decision[i] = 1 (* 1 means i included in solution M[i] *)
    else
        Decision[i] = 0 (* 0 means i not included in solution M[i] *)

S = ∅, i = n
While (i > 0) do
    if (Decision[i] == 1)
        S = S ∪ i
        i = p(i)
    else
        i = i-1

Output S
```