CS 473: Algorithms

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Part I

Exponentiation, Binary Search
Exponentiation

Input Two numbers: $a$ and integer $n \geq 0$

Goal Compute $a^n$
Exponentiation

Input  Two numbers: $a$ and integer $n \geq 0$

Goal  Compute $a^n$

Obvious algorithm:

\[
\text{SlowPow}(a,n): \\
\quad x = 1; \\
\quad \text{for } i = 1 \text{ to } n \text{ do} \\
\quad \quad x = x*a \\
\quad \text{Output } x
\]

$O(n)$ multiplications.
Fast Exponentiation

**Observation:** \( a^n = a^{\lfloor n/2 \rfloor} a^{n/2} = a^{\lfloor n/2 \rfloor} a^{n/2} a^{n/2} - \lfloor n/2 \rfloor}.\)
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FastPow(a,n):

- if \( n = 0 \) return 1
- \( x = \text{FastPow}(a, \lfloor n/2 \rfloor) \)
- \( x = x \times x \)
- if \( n \) is odd
  - \( x = x \times a \)
- return \( x \)
**Observation:** \( a^n = a^\lfloor n/2 \rfloor a^{\lceil n/2 \rceil} = a^\lfloor n/2 \rfloor a^{\lfloor n/2 \rfloor} a^{\lceil n/2 \rceil - \lfloor n/2 \rfloor} \).

**FastPow(a,n):**

1. if \( n = 0 \) return 1
2. \( x = \text{FastPow}(a,\lfloor n/2 \rfloor) \)
3. \( x = x*x \)
4. if \( n \) is odd
   1. \( x = x*a \)
5. return \( x \)

**T(n):** number of multiplications for \( n \)
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FastPow(a,n):
  if (n = 0) return 1
  x = FastPow(a,\lfloor n/2 \rfloor)
  x = x*x
  if (n is odd)
    x = x*a
  return x

$T(n)$: number of multiplications for $n$

$$T(n) \leq T(\lfloor n/2 \rfloor) + 2$$
Fast Exponentiation

Observation: \( a^n = a^{\lfloor n/2 \rfloor} a^{\lceil n/2 \rceil} = a^{\lfloor n/2 \rfloor} a^{\lceil n/2 \rceil} a^{\lfloor n/2 \rfloor - \lfloor n/2 \rfloor} \).

FastPow(a,n):

\[
\begin{align*}
    \text{if (n = 0) return 1} \\
    x &= \text{FastPow(a,}\lfloor n/2 \rfloor) \\
    x &= x \times x \\
    \text{if (n is odd)} \\
    \quad x &= x \times a \\
    \text{return x}
\end{align*}
\]

\( T(n) \): number of multiplications for \( n \)

\[ T(n) \leq T(\lfloor n/2 \rfloor) + 2 \]

\( T(n) = \Theta(\log n) \).
Complexity of Exponentiation

**Question:** Is SlowPow() a polynomial time algorithm? FastPow?
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Input size: $\log a + \log n$
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Input size: $\log a + \log n$

Output size:
Complexity of Exponentiation

**Question:** Is SlowPow() a polynomial time algorithm? FastPow?

Input size: \( \log a + \log n \)

Output size: \( n \log a \). Not necessarily polynomial in input size!

Both SlowPow and FastPow are polynomial in output size.
Exponentiation modulo a given number

Exponentiation in applications:

**Input**  Three integers: \( a, n \geq 0, \ p \geq 2 \) (typically a prime)

**Goal**  Compute \( a^n \ mod \ p \)
Exponentiation modulo a given number

Exponentiation in applications:

Input Three integers: $a, n \geq 0, p \geq 2$ (typically a prime)

Goal Compute $a^n \mod p$

Input size: $\Theta(\log a + \log n + \log p)$
Output size: $O(\log p)$ and hence polynomial in input size.
Exponentiation in applications:

**Input** Three integers: $a$, $n \geq 0$, $p \geq 2$ (typically a prime)

**Goal** Compute $a^n \mod p$

Input size: $\Theta(\log a + \log n + \log p)$
Output size: $O(\log p)$ and hence polynomial in input size.

**Observation:** $xy \mod p = ((x \mod p)(y \mod p)) \mod p$
Exponentiation modulo a given number

Input  Three integers:  $a, n \geq 0, p \geq 2$ (typically a prime)
Goal  Compute $a^n \mod p$

FastPowMod($a,n,p$):
  if ($n = 0$) return 1
  $x = \text{FastPowMod}(a,\lfloor n/2 \rfloor, p)$
  $x = x \times x \mod p$
  if ($n$ is odd)
      $x = x \times a \mod p$
  return $x$

FastPowMod is a polynomial time algorithm. SlowPowMod is not.
Exponentiation modulo a given number

**Input** Three integers: \( a, n \geq 0, p \geq 2 \) (typically a prime)

**Goal** Compute \( a^n \mod p \)

\[
\text{FastPowMod}(a,n,p):
\]
\[
\text{if } (n = 0) \text{ return } 1
\]
\[
x = \text{FastPowMod}(a,\lfloor n/2 \rfloor,p)
\]
\[
x = x \cdot x \mod p
\]
\[
\text{if } (n \text{ is odd})
\]
\[
x = x \cdot a \mod p
\]
\[
\text{return } x
\]

FastPowMod is a polynomial time algorithm. SlowPowMod is not (why?).
Binary Search in Sorted Arrays

**Input**  Sorted array \( A \) of \( n \) numbers and number \( x \)

**Goal**  Is \( x \) in \( A \)?

```plaintext
BinarySearch(A[a..b], x):
if (b-a <= 0) return NO
mid = A[⌊(a+b)/2⌋]
if (x = mid) return YES
else if (x < mid) return BinarySearch(A[a..⌊(a+b)/2⌋−1], x)
else return BinarySearch(A[⌊(a+b)/2⌋+1..b], x)
```

**Analysis:**

\[
T(n) = T(⌊n/2⌋) + O(1).
\]

\[
T(n) = O(\log n).
\]

**Observation:**
After \( k \) steps, size of array left is \( n/2^k \).
Binary Search in Sorted Arrays

Input Sorted array $A$ of $n$ numbers and number $x$

Goal Is $x$ in $A$?

BinarySearch($A[a..b]$, $x$):
  if ($b-a <= 0$) return NO
  mid = $A[\lfloor(a+b)/2\rfloor]$
  if ($x = mid$) return YES
  else if ($x < mid$) return BinarySearch($A[a..\lfloor(a+b)/2\rfloor-1], x$)
  else return BinarySearch($A[\lfloor(a+b)/2\rfloor+1..b], x$)

Analysis:
$T(n) = T(\lfloor n/2 \rfloor) + O(1)$.

Observation:
After $k$ steps, size of array left is $n/2^k$. 
Binary Search in Sorted Arrays

**Input**  Sorted array $A$ of $n$ numbers and number $x$

**Goal**  Is $x$ in $A$?

BinarySearch($A[a..b]$, $x$):

  if (b-a <= 0) return NO
  mid = $A[\lfloor (a+b)/2 \rfloor]$
  if (x = mid) return YES
  else if (x < mid) return BinarySearch($A[a..\lfloor (a+b)/2 \rfloor - 1], x$)
  else return BinarySearch($A[\lfloor (a+b)/2 \rfloor + 1..b], x$)

**Analysis:** $T(n) = T(\lfloor n/2 \rfloor) + O(1)$. $T(n) = O(\log n)$.

**Observation:** After $k$ steps, size of array left is $n/2^k$
Another common use of binary search

- **Optimization version**: find solution of best (say minimum) value
- **Decision version**: is there a solution of value at most a given value $v$?

Reduce optimization to decision (may be easier to think about):

- Given instance $I$ compute upper bound $U(I)$ on best value
- Compute lower bound $L(I)$ on best value
- Do binary search on interval $[L(I), U(I)]$ using decision version as black box

$O(\log(U(I) - L(I)))$ calls to decision version if $U(I), L(I)$ are integers
Another common use of binary search

- **Optimization version**: find solution of best (say minimum) value
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Reduce optimization to decision (may be easier to think about):

- Given instance $I$ compute upper bound $U(I)$ on best value
- Compute lower bound $L(I)$ on best value
- Do binary search on interval $[L(I), U(I)]$ using decision version as black box
- $O(\log(U(I) - L(I)))$ calls to decision version if $U(I), L(I)$ are integers
Example

- **Problem:** shortest paths in a graph.
- **Decision version:** given $G$ with non-negative integer edge lengths, nodes $s$, $t$ and bound $B$, is there an $s$-$t$ path in $G$ of length at most $B$?
- **Optimization version:** find the length of a shortest path between $s$ and $t$ in $G$.

**Question:** given a black box algorithm for the decision version, can we obtain an algorithm for the optimization version?
Example continued

**Question:** given a black box algorithm for the decision version, can we obtain an algorithm for the optimization version?

- Let $U$ be maximum edge length in $G$.
- Minimum edge length is $L$.
- $s$-$t$ shortest path length is at most $(n - 1)U$ and at least $L$.
- Apply binary search on the interval $[L, (n - 1)U]$ via the algorithm for the decision problem.
- $O(\log((n - 1)U - L))$ calls to the decision problem algorithm sufficient. Polynomial in input size.
Part II

Introduction to Dynamic Programming
Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction
  • reduce problem to a *smaller* instance of *itself*
  • self-reduction
Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction
- reduce problem to a *smaller* instance of *itself*
- self-reduction

Problem instance of size $n$ is reduced to one or more instances of size $n - 1$ or less.
- For termination, problem instances of small size are solved by some other method as *base cases*
Recursion in Algorithm Design

- **Tail Recursion**: problem reduced to a *single* recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms. Examples: Interval scheduling, MST algorithms, etc.

- **Divide and Conquer**: problem reduced to multiple *independent* sub-problems that are solved separately. Conquer step puts together solution for bigger problem.

- **Dynamic Programming**: problem reduced to multiple (typically) *dependent or overlapping* sub-problems. Use *memoization* to avoid recomputation of common solutions leading to *iterative bottom-up* algorithm.
Fibonacci Numbers

Fibonacci numbers defined by recurrence:

\[ F(n) = F(n - 1) + F(n - 2) \text{ and } F(0) = 0, F(1) = 1. \]

These numbers have many interesting and amazing properties. A journal *The Fibonacci Quarterly*!

- \[ F(n) = (\phi^n - (1 - \phi)^n)/\sqrt{5} \text{ where } \phi \text{ is the golden ratio } (1 + \sqrt{5})/2 \approx 1.618. \]
- \[ \lim_{n \to \infty} F(n + 1)/F(n) = \phi \]
Fibonacci Numbers

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These numbers have many interesting and amazing properties. A journal *The Fibonacci Quarterly*!

- \[ F(n) = \left( \phi^n - (1 - \phi)^n \right) / \sqrt{5} \] where \( \phi \) is the golden ratio \( (1 + \sqrt{5})/2 \approx 1.618. \)
- \( \lim_{n \to \infty} F(n+1)/F(n) = \phi \)

**Question:** Given \( n \), compute \( F(n) \).
Fibonacci Numbers

Recursive Algorithm for Fibonacci Numbers

Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n-1) + Fib(n-2)

Running time? Let $T(n)$ be the number of additions in Fib(n).

$T(n) = T(n−1) + T(n−2) + 1$ and $T(0) = T(1) = 0$

Roughly same as $F(n)$

$T(n) = \Theta(\phi^n)$

The number of additions is exponential in $n$.

Can we do better?
Fibonacci Numbers

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Fibonacci Numbers

Recursive Algorithm for Fibonacci Numbers

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$$T(n) = T(n - 1) + T(n - 2) + 1$$ and $T(0) = T(1) = 0$

Roughly same as $F(n)$

$$T(n) = \Theta(\phi^n)$$

The number of additions is exponential in $n$. Can we do better?
An iterative algorithm for Fibonacci numbers

Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        F[0] = 0
        F[1] = 1
        for i = 2 to n do
            F[i] = F[i-1] + F[i-2]
        return F[n]
An iterative algorithm for Fibonacci numbers

Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        F[0] = 0
        F[1] = 1
        for i = 2 to n do
            F[i] = F[i-1] + F[i-2]
        return F[n]

What is the running time of the algorithm?
Fibonacci Numbers

An iterative algorithm for Fibonacci numbers

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    if (n = 0)
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    else
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        F[1] = 1
        for i = 2 to n do
            F[i] = F[i-1] + F[i-2]
        return F[n]

What is the running time of the algorithm? $O(n)$ additions.
What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value.
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- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value. **Memoization.**

Dynamic Programming: finding a recursion that can be *effectively/efficiently* memoized

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?
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Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
        return Fib(n-1) + Fib(n-2)
Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

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How do we keep track of previously computed values?
Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

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  else
    return Fib(n-1) + Fib(n-2)

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Automatic explicit memoization

Initialize table/array \( M \) of size \( n \) such that \( M[i] = -1 \) for \( 0 \leq i < n \).
Automatic explicit memoization

Initialize table/array $M$ of size $n$ such that $M[i] = -1$ for $0 \leq i < n$

Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else if (M[n] \neq -1) (* M[n] has stored value of Fib(n) *)
        return M[n]
    else
        M[n] = Fib(n-1) + Fib(n-2)
        return M[n]

Need to know upfront the number of subproblems to allocate memory
Automatic implicit memoization

Initialize a (dynamic) dictionary data structure $D$ to empty

Fib(n):
  if (n = 0)
    return 0
  else if (n = 1)
    return 1
  else if (n is already in D)
    return value stored with n in D
  else
    val = Fib(n-1) + Fib(n-2)
    Store (n, val) in D
    return val
Explicit vs Implicit Memoization

- Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.
- Implicit and automatic memoization used when problem structure or algorithm is either not well understood or in fact unknown to the underlying system
  - need to pay overhead of datastructure
  - Functional languages such as LISP automatically do memoization, usually via hashing based dictionaries.
Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

Hence output size is exponential in input size so no polynomial time algorithm possible!

Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$.

Running time of recursive algorithm is $O(n\phi^n)$ but can in fact shown to be $O(\phi^n)$ by being careful. Doubly exponential in input size and exponential even in output size.
Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

- input is $n$ and hence input size is $\Theta(\log n)$
Is the iterative algorithm a \textit{polynomial} time algorithm? Does it take $O(n)$ time?

- Input is $n$ and hence input size is $\Theta(\log n)$
- Output is $F(n)$ and output size is $\Theta(n)$. Why?
Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

- input is $n$ and hence input size is $\Theta(\log n)$
- output is $F(n)$ and output size is $\Theta(n)$. Why?
- Hence output size is exponential in input size so no polynomial time algorithm possible!
Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

- Input is $n$ and hence input size is $\Theta(\log n)$
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- Hence output size is exponential in input size so no polynomial time algorithm possible!
- Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?
Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

- input is $n$ and hence input size is $\Theta(\log n)$
- output is $F(n)$ and output size is $\Theta(n)$. Why?
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Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?

Running time of recursive algorithm is $O(n\phi^n)$ but can in fact shown to be $O(\phi^n)$ by being careful. Doubly exponential in input size and exponential even in output size.
Part III

Brute Force Search, Recursion and Backtracking
Maximum Independent Set in a Graph

**Definition**

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an **independent set** (also called a stable set) if for there are no edges between nodes in $S$. That is, if $u, v \in S$ then $(u, v) \notin E$.

Some independent sets in graph above: $\{C, F\}$, $\{A, C, F\}$,
Maximum Independent Set Problem

**Input**  Graph $G = (V, E)$

**Goal**  Find maximum sized independent set in $G$
Input  Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal  Find maximum weight independent set in $G$
Maximum Weight Independent Set Problem

- No one knows an efficient (polynomial time) algorithm for this problem.
- Problem is NP-Complete and it is believed that there is no polynomial time algorithm.

A brute-force algorithm: try all subsets of vertices.
Brute-force enumeration

Algorithm to find the size of the maximum weight independent set.

MaxIndSet\((G = (V, E))\):
\[
\begin{align*}
max &= 0 \\
\text{for each subset } S &\subseteq V \\
&\quad \text{check if } S \text{ is an independent set} \\
&\quad \text{if } S \text{ is an independent set and } w(S) > max \\
&\quad \quad max = w(S) \\
&\quad \text{endfor} \\
\text{Output } max
\end{align*}
\]

Running time: suppose \(G\) has \(n\) vertices and \(m\) edges.
Checking each subset \(S\) takes \(O(m)\) time.
Total time is \(O(m^2 n)\).
Algorithm to find the size of the maximum weight independent set.

MaxIndSet\((G = (V, E))\):
\[
\begin{align*}
max & = 0 \\
\text{for each subset } & S \subseteq V \\
\text{check if } & S \text{ is an independent set} \\
\text{if } S & \text{ is an independent set and } w(S) > max \\
max & = w(S) \\
\text{endfor} \\
\text{Output } & max
\end{align*}
\]

Running time: suppose \(G\) has \(n\) vertices and \(m\) edges
\[
\begin{itemize}
\item 2^n \text{ subsets of } V \\
\item \text{checking each subset } S \text{ takes } O(m) \text{ time} \\
\item \text{total time is } O(m2^n)
\end{itemize}
\]
Let $V = \{v_1, v_2, \ldots, v_n\}$.
For a vertex $u$ let $N(u)$ be its neighbours.
A Recursive Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$. For a vertex $u$ let $N(u)$ be its neighbours.

Observation

One of the following two cases is true

Case 1 \( v_n \) is in some maximum independent set.
Case 2 \( v_n \) is in no maximum independent set.
A Recursive Algorithm

Let \( V = \{v_1, v_2, \ldots, v_n\} \).

For a vertex \( u \) let \( N(u) \) be its neighbours.

**Observation**

One of the following two cases is true

- **Case 1** \( v_n \) is in some maximum independent set.
- **Case 2** \( v_n \) is in no maximum independent set.

**Recursive-MIS**(\( G \)):

- If \( G \) is empty, Output 0
- \( a = \text{Recursive-MIS}(G - v_n) \)
- \( b = w(v_n) + \text{Recursive-MIS}(G - v_n - N(v_n)) \)
- Output \( \max(a, b) \)
Recursive Algorithms of MIS

Running time:

\[ T(n) = \]

where \( \deg(v_1) \) is the degree of \( v_1 \).

\[ T(0) = T(1) = 1 \] is base case.

Worst case is when \( \deg(v_1) = 0 \) when the recurrence becomes

\[ T(n) = 2T(n-1) + O(1) \]

Solution to this is \( T(n) = O(2^n) \).
Recursive Algorithms of MIS

Running time:

\[ T(n) = T(n - 1) + T(n - 1 - \text{deg}(v_1)) + O(1) \]

where \( \text{deg}(v_1) \) is the degree of \( v_1 \). \( T(0) = T(1) = 1 \) is base case.

Worst case is when \( \text{deg}(v_1) = 0 \) when the recurrence becomes

\[ T(n) = 2T(n - 1) + O(1) \]

Solution to this is \( T(n) = O(2^n) \).
Backtrack Search via Recursion

- Recursive algorithm generates a tree of computation where each node is a smaller problem (subproblem).
- Simple recursive algorithm computes/explores the whole tree blindly in some order.
- Backtrack search is a way to explore the tree intelligently to prune the search space.
  - Some subproblems may be so simple that we can stop the recursive algorithm and solve it directly by some other method.
  - Memoization to avoid recomputing same problem.
  - Stop recursing at a subproblem if it is clear that there is no need to explore further.
- Leads to a number of heuristics that are widely used in practice although the worst case running time may still be exponential.
Example