Part I

Breadth First Search
Overview

- BFS is obtained from BasicSearch by processing edges using a data structure called a queue.
- It processes the vertices in the graph in the order of their shortest distance from the vertex $s$ (the start vertex).

DFS good for exploring graph structure

BFS good for exploring distances
A queue is a list of elements which supports the following operations.
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- **enqueue**: Adds an element to the end of the list
Queue Data Structure

Queues

A *queue* is a list of elements which supports the following operations

- **enqueue**: Adds an element to the end of the list
- **dequeue**: Removes an element from the front of the list

Elements are extracted in first-in first-out (FIFO) order, i.e., elements are picked in the order in which they were inserted.
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BFS Algorithm

Given (undirected or directed) graph \( G = (V, E) \) and node \( s \in V \)

BFS(s)
- Mark all vertices as unvisited
- Initialize search tree \( T \) to be empty
- Mark vertex \( s \) as visited
- set \( Q \) to be the empty queue
- \( \text{enq}(s) \)
- while \( Q \) is nonempty
  - \( u = \text{deq}(Q) \)
  - for each vertex \( v \) in Adj\((u)\)
    - if \( v \) is not visited
      - add edge \((u,v)\) to \( T \)
      - Mark \( v \) as visited and \( \text{enq}(v) \)
BFS Algorithm

Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

BFS(s)

Mark all vertices as unvisited
Initialize search tree $T$ to be empty
Mark vertex $s$ as visited
set $Q$ to be the empty queue
enq(s)
while $Q$ is nonempty
  $u = \text{deq}(Q)$
  for each vertex $v$ in Adj($u$)
    if $v$ is not visited
      add edge $(u,v)$ to $T$
      Mark $v$ as visited and enq(v)

Proposition

$BFS(s)$ runs in $O(n + m)$ time.
1. \([1]\)
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]

BFS tree is the set of black edges.
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]
BFS: An Example in Undirected Graphs

1. [1]  
2. [2,3]  
3. [3,4,5]  

4. [4,5,7,8]  

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1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
BFS: An Example in Undirected Graphs

1. [1] 4. [4,5,7,8]
2. [2,3] 5. [5,7,8]
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3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

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BFS: An Example in Directed Graphs

Definition

A directed graph (also called a digraph) is \( G = (V, E) \), where

- \( V \) is a set of vertices or nodes
- \( E \subseteq V \times V \) is the set of ordered pairs of vertices called edges
BFS: An Example in Directed Graphs

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**Viswanathan CS473**
BFS(s)
Mark all vertices as unvisited and for each v set dist(v) = ∞
Initialize search tree $T$ to be empty
Mark vertex s as visited and set dist(s) = 0
set Q to be the empty queue
enq(s)
while Q is nonempty
    u = deq(Q)
    for each vertex v in Adj(u)
        if v is not visited
            add edge (u, v) to T
            Mark v as visited, enq(v)
            and set dist(v) = dist(u) + 1
Properties of BFS: Undirected Graphs

Proposition

*The following properties hold upon termination of BFS(s)*

- The search tree contains exactly the set of vertices in the connected component of s.
- If \( \text{dist}(u) < \text{dist}(v) \) then u is visited before v.
- For every vertex u, \( \text{dist}(u) \) is indeed the length of shortest path from s to u.
- If u, v are in connected component of s and e = \{u, v\} is an edge of G, then either e is an edge in the search tree, or \(|\text{dist}(u) - \text{dist}(v)| \leq 1\).

Proof. Exercise.
Properties of BFS: Undirected Graphs

**Proposition**

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**Proof.**

Exercise.
Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of BFS(s):

- The search tree contains exactly the set of vertices reachable from s.
- If \( \text{dist}(u) < \text{dist}(v) \), then u is visited before v.
- For every vertex u, \( \text{dist}(u) \) is indeed the length of shortest path from s to u.
- If \( u \) is reachable from s and \( e = (u, v) \) is an edge of G, then either \( e \) is an edge in the search tree, or \( \text{dist}(v) - \text{dist}(u) \leq 1 \). Not necessarily the case that \( \text{dist}(u) - \text{dist}(v) \leq 1 \).

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- The search tree contains exactly the set of vertices reachable from $s$.
- If $\text{dist}(u) < \text{dist}(v)$ then $u$ is visited before $v$.
Properties of BFS: Directed Graphs

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The following properties hold upon termination of BFS(s):

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Proof.

Exercise.
BFS-Layers(s):
Mark all vertices as unvisited and initialize T to be empty
Mark s as visited and set $L_0 = \{s\}$

$i = 0$

While $L_i$ is not empty do
    initialize $L_{i+1}$ to be an empty list
    for each $u$ in $L_i$ do
        for each edge $(u,v)$ in Adj($u$) do
            if $v$ is not visited
                mark $v$ as visited
                add $(u,v)$ to tree $T$
                add $v$ to $L_{i+1}$

    $i = i + 1$
BFS-Layers(s):
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                add $(u,v)$ to tree $T$
                add $v$ to $L_{i+1}$
        $i = i + 1$

Running time: $O(n + m)$
Example
BFS with Layers: Properties

Proposition

The following properties hold on termination of BFS-Layers(s).

- **BFS-Layers(s) outputs a BFS tree**
- **$L_i$ is the set of vertices at distance exactly $i$ from $s$**
- **If $G$ is undirected, each edge $e = \{u, v\}$ is one of three types:**
  - tree edge between two consecutive layers
  - non-tree forward/backward edge between two consecutive layers
  - non-tree cross-edge with both $u, v$ in same layer
- **If $G$ is directed, each edge $e = (u, v)$ is one of four types:**
  - a tree edge between consecutive layers, $u \in L_i, v \in L_{i+1}$ for some $i \geq 0$
  - a non-tree forward edge between consecutive layers
  - a non-tree backward edge
  - a cross-edge with both $u, v$ in same layer
Part II

Bipartite Graphs and an application of BFS
Bipartite Graphs

Definition (Bipartite Graph)

Undirected graph \( G = (V, E) \) is a bipartite graph if \( V \) can be partitioned into \( X \) and \( Y \) such that all edges in \( E \) are between \( X \) and \( Y \).
Question
When is a graph bipartite?

Proposition
Every tree is a bipartite graph.

Proof.
Root tree $T$ at some node $r$. Let $L_i$ be all nodes at level $i$, that is, $L_i$ is all nodes at distance $i$ from root $r$. Now define $X$ to be all nodes at even levels and $Y$ to be all nodes at odd level. Only edges in $T$ are between levels.

Proposition
An odd length cycle is not bipartite.
Bipartite Graph Characterization

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Proposition

*An odd length cycle is not bipartite.*
Proposition

An odd length cycle is not bipartite.

Proof.

Let $C = u_1, u_2, \ldots, u_{2k+1}, u_1$ be an odd cycle. Suppose $C$ is a bipartite graph and let $X, Y$ be the bipartition. Without loss of generality $u_1 \in X$. Implies $u_2 \in Y$. Implies $u_3 \in X$. Inductively, $u_i \in X$ if $i$ is odd $u_i \in Y$ if $i$ is even. But $\{u_1, u_{2k+1}\}$ is an edge and both belong to $X$!
**Definition**

Given a graph $G = (V, E)$ a **subgraph** of $G$ is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. 

---

**Proposition**

If $G$ is bipartite then any subgraph $H$ of $G$ is also bipartite.

**Proposition**

A graph $G$ is not bipartite if $G$ has an odd cycle $C$ as a subgraph.

**Proof.**

If $G$ is bipartite then since $C$ is a subgraph, $C$ is also bipartite (by above proposition). However, $C$ is not bipartite!
Subgraphs

Definition
Given a graph $G = (V, E)$ a subgraph of $G$ is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

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Subgraphs

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If $G$ is bipartite then any subgraph $H$ of $G$ is also bipartite.

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Subgraphs

Definition
Given a graph \( G = (V, E) \) a subgraph of \( G \) is another graph \( H = (V', E') \) where \( V' \subseteq V \) and \( E' \subseteq E \).

Proposition
If \( G \) is bipartite then any subgraph \( H \) of \( G \) is also bipartite.

Proposition
A graph \( G \) is not bipartite if \( G \) has an odd cycle \( C \) as a subgraph.

Proof.
If \( G \) is bipartite then since \( C \) is a subgraph, \( C \) is also bipartite (by above proposition). However, \( C \) is not bipartite!
### Theorem

A graph $G$ is bipartite if and only if it has no odd length cycle as subgraph.

### Proof.

**Only If:** $G$ has an odd cycle implies $G$ is not bipartite.
Bipartite Graph Characterization

**Theorem**

A graph $G$ is bipartite if and only if it has no odd length cycle as subgraph.

**Proof.**

*Only If:* $G$ has an odd cycle implies $G$ is not bipartite.

*If:* $G$ has no odd length cycle. Assume without loss of generality that $G$ is connected.

- Pick $u$ arbitrarily and do BFS($u$)
- $X = \bigcup_{i \text{ is even}} L_i$ and $Y = \bigcup_{i \text{ is odd}} L_i$

**Claim:** $X$ and $Y$ is a valid bipartition if $G$ has no odd length cycle.
Proof of Claim

Claim

In BFS(u) if a, b ∈ L; and (a, b) is an edge then there is an odd length cycle containing (a, b).

Proof.

Let v be least common ancestor of a, b in BFS tree T. v is in some level j < i (could be u itself). Path from v ⇝ a in T is of length j − i. Path from v ⇝ b in T is of length j − i. These two paths plus plus (a, b) forms an odd cycle of length 2(j − i) + 1.

Corollary

There is an O(n + m) time algorithm to check if G is bipartite and output an odd cycle if it is not.
Proof of Claim

Claim

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Proof.

Let $v$ be least common ancestor of $a, b$ in BFS tree $T$. $v$ is in some level $j < i$ (could be $u$ itself).
Path from $v \leadsto a$ in $T$ is of length $j - i$.
Path from $v \leadsto b$ in $T$ is of length $j - i$.
These two paths plus plus $(a, b)$ forms an odd cycle of length $2(j - i) + 1$. □

Corollary

There is an $O(n + m)$ time algorithm to check if $G$ is bipartite and output an odd cycle if it is not.
Proof of Claim

Claim

In BFS(u) if a, b ∈ L; and (a, b) is an edge then there is an odd length cycle containing (a, b).

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These two paths plus plus (a, b) forms an odd cycle of length 2(j – i) + 1.

Corollary

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Part III

Shortest Paths and Dijkstra’s Algorithm
Shortest Path Problems

**Input**  A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s$, $t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.
- Find shortest paths for all pairs of nodes.
Shortest Path Problems

Input  A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.
- Find shortest paths for all pairs of nodes.

Many applications!
Single-Source Shortest Path Problems

**Input** A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.
Single-Source Shortest Paths: Non-Negative Edge Lengths

**Single-Source Shortest Path Problems**

**Input**
A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.

- Restrict attention to directed graphs
- Undirected graph problem can be reduced to directed graph problem - how?
Single-Source Shortest Paths: Non-Negative Edge Lengths

Single-Source Shortest Path Problems

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- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.

- Restrict attention to directed graphs
- Undirected graph problem can be reduced to directed graph problem - how?
  - Given undirected graph $G$, create a new directed graph $G'$ by replacing each edge $\{u, v\}$ in $G$ by $(u, v)$ and $(v, u)$ in $G'$.
  - set $\ell(u, v) = \ell(v, u) = \ell(\{u, v\})$
  - Exercise: show reduction works
Single-Source Shortest Paths via BFS

**Special case:** All edge lengths are 1.
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- Run BFS(s) to get shortest path distances from s to all other nodes.
- $O(m + n)$ time algorithm.
Single-Source Shortest Paths via BFS

**Special case:** All edge lengths are 1.

- Run BFS(s) to get shortest path distances from s to all other nodes.
- \( O(m + n) \) time algorithm.

**Special case:** Suppose \( \ell(e) \) is an integer for all \( e \)? Can we use BFS?
Single-Source Shortest Paths via BFS

**Special case:** All edge lengths are 1.
- Run BFS(s) to get shortest path distances from s to all other nodes.
- $O(m + n)$ time algorithm.

**Special case:** Suppose $\ell(e)$ is an integer for all $e$?
Can we use BFS? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on $e$.
**Special case:** All edge lengths are 1.
- Run BFS(s) to get shortest path distances from s to all other nodes.
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**Special case:** Suppose \(\ell(e)\) is an integer for all \(e\)? Can we use BFS? Reduce to unit edge-length problem by placing \(\ell(e) - 1\) dummy nodes on \(e\)

Let \(L = \max_e \ell(e)\). New graph has \(O(mL)\) edges and \(O(mL + n)\) nodes. BFS takes \(O(mL + n)\) time. Not efficient if \(L\) is large.
Towards an algorithm

Why does BFS work?

Lemma

Let G be a directed graph with non-negative edge lengths. Let \( \text{dist}(s, v) \) denote the shortest path length from \( s \) to \( v \). If \( s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \) is a shortest path from \( s \) to \( v \) then for \( 1 \leq i < k \):

\[
\text{dist}(s, v_i) \leq \text{dist}(s, v_k).
\]

Proof.

Suppose not. Then for some \( i < k \) there is a path \( P' \) from \( s \) to \( v_i \) of length strictly less than that of \( s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i \). Then \( P' \) concatenated with \( v_i \rightarrow v_{i+1} \rightarrow \ldots \rightarrow v_k \) contains a strictly shorter path to \( v \) than \( s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_k \).
Towards an algorithm

Why does BFS work?

BFS(s) explores nodes in increasing distance from s

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- $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$
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Towards an algorithm

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- $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$
- $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$.

Proof.

Suppose not. Then for some $i < k$ there is a path $P'$ from $s$ to $v_i$ of length strictly less than that of $s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$. Then $P'$ concatenated with $v_i \rightarrow v_{i+1} \ldots \rightarrow v_k$ contains a strictly shorter path to $v_k$ than $s = v_0 \rightarrow v_1 \ldots \rightarrow v_k$. 

A Basic Strategy

Explore vertices in increasing order of distance from $s$:
(For simplicity assume that nodes are at different distances from $s$
and that no edge has zero length)

Initialize for each node $v$, $\text{dist}(s,v) = \infty$
Initialize $S = \emptyset$,
for $i = 1$ to $|V|$ do
  (* Invariant: $S$ contains the $i$-1 closest nodes to $s$ *)
  Among nodes in $V-S$, find the node $v$ that is the $i$'th closest to $s$
  Update $\text{dist}(s,v)$
  $S = S \cup \{v\}$
Explore vertices in increasing order of distance from $s$:
(For simplicity assume that nodes are at different distances from $s$ and that no edge has zero length)

Initialize for each node $v$, $\text{dist}(s,v) = \infty$
Initialize $S = \emptyset$,
for $i = 1$ to $|V|$ do
  (* Invariant: $S$ contains the $i$-1 closest nodes to $s$ *)
  Among nodes in $V-S$, find the node $v$ that is the $i$’th closest to $s$
  Update $\text{dist}(s,v)$
  $S = S \cup \{v\}$

How can we implement the step in the for loop?
Finding the $i$’th closest node

- $S$ contains the $i−1$ closest nodes to $s$
- Want to find the $i$’th closest node from $V − S$.

What do we know about the $i$’th closest node?
Finding the $i$’th closest node

- $S$ contains the $i - 1$ closest nodes to $s$
- Want to find the $i$’th closest node from $V - S$.

What do we know about the $i$’th closest node?

**Claim**

*Let $P$ be a shortest path from $s$ to $v$ where $v$ is the $i$’th closest node. Then, all intermediate nodes in $P$ belong to $S$.***
Finding the $i$’th closest node

- $S$ contains the $i – 1$ closest nodes to $s$
- Want to find the $i$’th closest node from $V – S$.

What do we know about the $i$’th closest node?

**Claim**

*Let $P$ be a shortest path from $s$ to $v$ where $v$ is the $i$’th closest node. Then, all intermediate nodes in $P$ belong to $S$.***

**Proof.**

If $P$ had an intermediate node $u$ not in $S$ then $u$ will be closer to $s$ than $v$. Implies $v$ is not the $i$’th closest node to $s$ - recall that $S$ already has the $i – 1$ closest nodes.
Finding the $i$’th closest node

Corollary

*The $i$’th closest node is adjacent to $S$.*
Finding the $i$’th closest node

- $S$ contains the $i - 1$ closest nodes to $s$
- Want to find the $i$’th closest node from $V - S$.

For each $u \in V - S$ let $P(s, u, S)$ be a shortest path from $s$ to $u$ using only nodes in $S$ as intermediate vertices.

Let $d'(s, u)$ be the length of $P(s, u, S)$
Finding the $i$’th closest node

- $S$ contains the $i - 1$ closest nodes to $s$
- Want to find the $i$’th closest node from $V - S$.

- For each $u \in V - S$ let $P(s, u, S)$ be a shortest path from $s$ to $u$ using only nodes in $S$ as intermediate vertices.
- Let $d'(s, u)$ be the length of $P(s, u, S)$

Observations: for each $u \in V - S$,

- $\text{dist}(s, u) \leq d'(s, u)$ since we are constraining the paths
- $d'(s, u) = \min_{a \in S}(\text{dist}(s, a) + \ell(a, u))$ - Why?
Finding the $i$’th closest node

- $S$ contains the $i - 1$ closest nodes to $s$
- Want to find the $i$’th closest node from $V - S$.

For each $u \in V - S$ let $P(s, u, S)$ be a shortest path from $s$ to $u$ using only nodes in $S$ as intermediate vertices.
- Let $d'(s, u)$ be the length of $P(s, u, S)$

Observations: for each $u \in V - S$,

- $dist(s, u) \leq d'(s, u)$ since we are constraining the paths
- $d'(s, u) = \min_{a \in S} (dist(s, a) + \ell(a, u))$ - Why?

**Lemma**

If $\nu$ is the $i$’th closest node to $s$, then $d'(s, \nu) = dist(s, \nu)$. 
Finding the $i$’th closest node

Lemma

If $v$ is an $i$’th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$.

Proof.

Let $v$ be the $i$’th closest node to $s$. Then there is a shortest path $P$ from $s$ to $v$ that contains only nodes in $S$ as intermediate nodes (see prev claim). Therefore $d'(s, v) = \text{dist}(s, v)$.  \[\square\]
Finding the $i$’th closest node

**Lemma**

If $v$ is an $i$’th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$.

**Corollary**

The $i$’th closest node to $s$ is the node $v \in V - S$ such that $d'(s, v) = \min_{u \in V - S} d'(s, u)$.

**Proof.**

For every node $u \in V - S$, $\text{dist}(s, u) \leq d'(s, u)$ and for the $i$’th closest node $v$, $\text{dist}(s, v) = d'(s, v)$. Moreover, $\text{dist}(s, u) \geq \text{dist}(s, v)$ for each $u \in V - S$. 

Algorithm

Initialize for each node \( v \), \( \text{dist}(s,v) = \infty \)
Initialize \( S = \emptyset \), \( d'(s,s) = 0 \)
for \( i = 1 \) to \( |V| \) do

(* Invariant: \( S \) contains the \( i-1 \) closest nodes to \( s \) *)
(* Invariant: \( d'(s,u) \) is shortest path distance from \( u \) to \( s \) using only \( S \) as intermediate nodes*)

Let \( v \) be such that \( d'(s,v) = \min_{u \in V-S} d'(s,u) \)
\( \text{dist}(s,v) = d'(s,v) \)
\( S = S \cup \{v\} \)
for each node \( u \) in \( V-S \)

compute \( d'(s,u) = \min_a \in S (\text{dist}(s,a) + \ell(a,u)) \)

endfor
**Algorithm**

Initialize for each node $v$, $\text{dist}(s,v) = \infty$
Initialize $S = \emptyset$, $d'(s,s) = 0$
for $i = 1$ to $|V|$ do

(* Invariant: $S$ contains the $i-1$ closest nodes to $s$ *)
(* Invariant: $d'(s,u)$ is shortest path distance from $u$ to $s$
  using only $S$ as intermediate nodes*)
Let $v$ be such that $d'(s,v) = \min_{u \in V-S} d'(s,u)$
$\text{dist}(s,v) = d'(s,v)$
$S = S \cup \{v\}$
for each node $u$ in $V$–$S$

compute $d'(s,u) = \min_{a \in S} (\text{dist}(s,a) + \ell(a,u))$
endfor

**Correctness:** By induction on $i$ using previous lemmas.
Algorithm

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Initialize $S = \emptyset$, $d'(s,s) = 0$
for $i = 1$ to $|V|$ do
    (* Invariant: $S$ contains the $i-1$ closest nodes to $s$ *)
    (* Invariant: $d'(s,u)$ is shortest path distance from $u$ to $s$
    using only $S$ as intermediate nodes*)
    Let $v$ be such that $d'(s,v) = \min_{u \in V - S} d'(s,u)$
    $\text{dist}(s,v) = d'(s,v)$
    $S = S \cup \{v\}$
    for each node $u$ in $V - S$
        compute $d'(s,u) = \min_{a \in S} (\text{dist}(s,a) + \ell(a,u))$
    endfor

Correctness: By induction on $i$ using previous lemmas.
Running time:
Priority Queues

Algorithm

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  (* Invariant: $S$ contains the $i$ closest nodes to $s$ *)
  (* Invariant: $d'(s,u)$ is shortest path distance from $u$ to $s$
    using only $S$ as intermediate nodes*)
  Let $v$ be such that $d'(s,v) = \min_{u \in V-S} d'(s,u)$
  $\text{dist}(s,v) = d'(s,v)$
  $S = S \cup \{v\}$
  for each node $u$ in $V-S$
    compute $d'(s,u) = \min_{a \in S} (\text{dist}(s,a) + \ell(a,u))$
  endfor

Correctness: By induction on $i$ using previous lemmas.
Running time: $O(n \cdot (n + m))$ time.

- $n$ outer iterations. In each iteration, $d'(s,u)$ for each $u$ by
  scanning all edges out of nodes in $S$; $O(m + n)$ time/iteration.
Example
Example
Example
Example

Priority Queues

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Example
Example
Priority Queues

Example

Diagram of a network with weighted edges and nodes labeled with values.
Example
Improved Algorithm

- Main work is to compute the $d'(s, u)$ values in each iteration.
- $d'(s, u)$ changes from iteration $i$ to $i + 1$ only because of the node $v$ that is added to $S$ in iteration $i$. 

```plaintext
Initialize for each node $v$, dist(s,v) = d'(s,v) = \infty
Initialize S = \emptyset, d'(s,s) = 0
for i = 1 to |V| do
    (*S contains the i-1 closest nodes to s, d'(s,u) values current *)
    Let v be such that d'(s,v) = min_{u \in V - S} d'(s,u)
    dist(s,v) = d'(s,v)
    S = S \cup \{ v \}
    Update d'(s,u) for each u in V-S as follows:
    d'(s,u) = min (d'(s,u), dist(s,v) + \ell(v,u))

Running time: \(O(m + n^2)\) time.

Finding $v$ from $d'(s,u)$ values is \(O(n)\) time.
```
Improved Algorithm

- Main work is to compute the $d'(s, u)$ values in each iteration
- $d'(s, u)$ changes from iteration $i$ to $i + 1$ only because of the node $v$ that is added to $S$ in iteration $i$.

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  Let $v$ be such that $d'(s,v) = \min_{u \in V-S} d'(s,u)$
  $\text{dist}(s,v) = d'(s,v)$
  $S = S \cup \{v\}$
  Update $d'(s,u)$ for each $u$ in $V-S$ as follows:
    $d'(s,u) = \min (d'(s,u), \text{dist}(s,v) + \ell(v,u))$

Running time:
Improved Algorithm

- Main work is to compute the $d'(s, u)$ values in each iteration.
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  $\text{dist}(s,v) = d'(s,v)$
  $S = S \cup \{v\}$
  Update $d'(s,u)$ for each $u$ in $V-S$ as follows:
  $d'(s,u) = \min (d'(s,u), \text{dist}(s,v) + \ell(v,u))$

Running time: $O(m + n^2)$ time.
- $n$ outer iterations and in each iteration following steps
- updating $d'(s, u)$ after $v$ added takes $O(\text{deg}(v))$ time so total work is $O(m)$ since a node enters $S$ only once
- Finding $v$ from $d'(s, u)$ values is $O(n)$ time
Dijkstra’s Algorithm

- eliminate \( d'(s, u) \) and let \( \text{dist}(s, u) \) maintain it
- update \( \text{dist} \) values after adding \( v \) by scanning edges out of \( v \)

Initialize for each node \( v \), \( \text{dist}(s, v) = \infty \)
Initialize \( S = \{s\} \), \( \text{dist}(s, s) = 0 \)
for \( i = 1 \) to \(|V|\) do
  Let \( v \) be such that \( \text{dist}(s, v) = \min_{u \in V - S} \text{dist}(s, u) \)
  \( S = S \cup \{v\} \)
  For each \( u \) in \( \text{Adj}(v) \) do
    \( \text{dist}(s, u) = \min(\text{dist}(s, u), \text{dist}(s, v) + \ell(v,u)) \)

Priority Queues to maintain \( \text{dist} \) values for faster running time
Dijkstra’s Algorithm

- eliminate \(d'(s, u)\) and let \(dist(s, u)\) maintain it
- update \(dist\) values after adding \(v\) by scanning edges out of \(v\)

Initialize for each node \(v\), \(dist(s,v) = \infty\)
Initialize \(S = \{s\}\), \(dist(s,s) = 0\)
for \(i = 1\) to \(|V|\) do
  Let \(v\) be such that \(dist(s,v) = \min_{u \in V - S} dist(s,u)\)
  \(S = S \cup \{v\}\)
  For each \(u\) in \(\text{Adj}(v)\) do
    \(dist(s,u) = \min (dist(s,u), dist(s,v) + \ell(v,u))\)

Priority Queues to maintain \(dist\) values for faster running time
- Using heaps and standard priority queues: \(O((m + n)\log n)\)
- Using Fibonacci heaps: \(O(m + n\log n)\).
Priority Queues

Data structure to store a set $S$ of $n$ elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations

- **makeQ**: create an empty queue
- **findMin**: find the minimum key in $S$
- **extractMin**: Remove $v \in S$ with smallest key and return it
- **add($v$, $k(v)$)**: Add new element $v$ with key $k(v)$ to $S$
- **delete($v$)**: Remove element $v$ from $S$

decreaseKey($v$, $k'(v)$): decrease key of $v$ from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$

meld: merge two separate priority queues into one can be performed in $O(\log n)$ time each.

decreaseKey via delete and add
Priority Queues

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Priority Queues

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- **add\((v, k(v))\)**: Add new element \( v \) with key \( k(v) \) to \( S \)
- **delete\((v)\)**: Remove element \( v \) from \( S \)
- **decreaseKey\((v, k'(v))\)**: *decrease* key of \( v \) from \( k(v) \) (current key) to \( k'(v) \) (new key). Assumption: \( k'(v) \leq k(v) \)
- **meld**: merge two separate priority queues into one

can be performed in \( O(\log n) \) time each.

decreaseKey via delete and add
Dijkstra’s Algorithm using Priority Queues

Q = makePQ()
insert(Q, (s,0))
for each node u ≠ s
    insert(Q, (u,∞))
S = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    S = S ∪ {v}
    For each u in Adj(v) do
        decreaseKey(Q, (u, min (dist(s,u), dist(s,v) + ℓ(v,u))))

Priority Queue operations:
- \(O(n)\) insert operations
- \(O(n)\) extractMin operations
- \(O(m)\) decreaseKey operations
Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

- All operations can be done in $O(\log n)$ time
Implementing Priority Queues via Heaps

Using Heaps

- Store elements in a heap based on the key value
- All operations can be done in $O(\log n)$ time

Dijkstra’s algorithm can be implemented in $O((n + m) \log n)$ time.
**Fibonacci Heaps**

- `extractMin`, `add`, `delete`, `meld` in $O(\log n)$ time
- `decreaseKey` in $O(1)$ *amortized* time.
Fibonacci Heaps

- extractMin, add, delete, meld in $O(\log n)$ time
- decreaseKey in $O(1)$ amortized time: $\ell$ decreaseKey operations for $\ell \geq n$ take together $O(\ell)$ time
- Relaxed Heaps: decreaseKey in $O(1)$ worst case time but at the expense of meld (not necessary for Dijkstra’s algorithm)

Dijkstra’s algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.

Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)
Priority Queues via Fibonacci Heaps and Relaxed Heaps

**Fibonacci Heaps**
- extractMin, add, delete, meld in $O(\log n)$ time
- decreaseKey in $O(1)$ *amortized* time: $\ell$ decreaseKey operations for $\ell \geq n$ take *together* $O(\ell)$ time
- Relaxed Heaps: decreaseKey in $O(1)$ worst case time but at the expense of meld (not necessary for Dijkstra’s algorithm)

Dijkstra’s algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
Fibonacci Heaps

- extractMin, add, delete, meld in $O(\log n)$ time
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Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)
Dijkstra’s algorithm finds the shortest path distances from $s$ to $V$.

**Question:** How do we find the paths themselves?
Dijkstra’s algorithm finds the shortest path distances from s to V. **Question:** How do we find the paths themselves?

Q = makePQ()
insert(Q, (s,0))
prev(s) = null
for each node u \neq s
    insert(Q, (u,\infty))
    prev(u) = null

S = \emptyset
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    S = S \cup \{v\}
    For each u in Adj(v) do
        if (dist(s,v) + \ell(v,u) < dist(s,u)) then
            decreaseKey(Q, (u, dist(s,v) + \ell(v,u)))
            prev(u) = v
**Lemma**

The edge set $(u, \text{prev}(u))$ is the reverse of a shortest path tree rooted at $s$. For each $u$, the reverse of the path from $u$ to $s$ in the tree is a shortest path from $s$ to $u$.

**Proof Sketch.**

- The edgeset $\{(u, \text{prev}(u)) \mid u \in V\}$ induces a directed in-tree rooted at $s$ (Why?)
- Use induction on $|S|$ to argue that the tree is a shortest path tree for nodes in $V$. 
Dijkstra’s algorithm gives shortest paths from $s$ to all nodes in $V$.

How do we find shortest paths from all of $V$ to $s$?
Shortest paths to $s$

Dijkstra’s algorithm gives shortest paths from $s$ to all nodes in $V$.

How do we find shortest paths from all of $V$ to $s$?

- In undirected graphs shortest path from $s$ to $u$ is a shortest path from $u$ to $s$ so there is no need to distinguish.
- In directed graphs, use Dijkstra’s algorithm in $G^{\text{rev}}$!
Takeaway Points

- BFS is a search strategy that explores the graph in increasing order of distance from the start vertex. BFS layers are useful structural information when thinking about distances.
- Shortest path problems are ubiquitous in applications. Single-source shortest paths for non-negative edge lengths can be computed efficiently via Dijkstra’s algorithm. The algorithms relies on principles similar to BFS, that is, the vertices can be found in increasing order of distance from the source. Make sure you understand why non-negative edge lengths are important for this.
- The shortest path distances, and also paths, from a given source can be compactly represented via a shortest path tree.
- Shortest path distances are asymmetric in directed graphs and distances to a source vertex can be computed by running the algorithm in reverse of the original graph.