CS 473: Algorithms

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Recap

NP: languages that have polynomial time certifiers/verifiers

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A language L is NP-Complete iff

- L is in NP
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Theorem (Cook-Levin)

Circuit-SAT and SAT are NP-Complete.

Recap contd

Theorem (Cook-Levin)

Circuit-SAT and SAT are NP-Complete.

Establish NP-Completeness via reductions:

- SAT $\leq_P$ 3-SAT and hence 3-SAT is NP-complete
- 3-SAT $\leq_P$ Independent Set (which is in NP) and hence Independent Set is NP-complete
- Vertex Cover is NP-complete
- Clique is NP-complete
- Set Cover is NP-Complete

Today

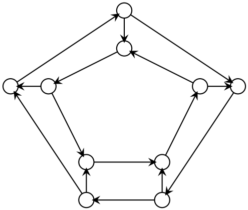
Prove

- Hamiltonian Cycle Problem is NP-Complete
- 3-Coloring is NP-Complete

Directed Hamiltonian Cycle

Input Given a directed graph $G = (V, E)$ with n vertices

Goal Does G have a Hamiltonian cycle?



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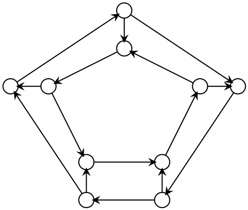

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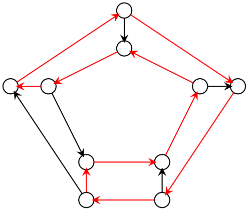

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- Hardness: We will show
  3-SAT $\leq_P$ Directed Hamiltonian Cycle

Reduction

Given 3-SAT formula φ create a graph G_φ such that

- G_φ has a Hamiltonian cycle if and only if φ is satisfiable
- G_φ should be constructible from φ by a polynomial time algorithm $\mathcal{A}$

Notation: φ has n variables $x_1, x_2, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$.

Reduction: First Ideas

- Viewing SAT: Assign values to n variables, and each clauses has 3 ways in which it can be satisfied

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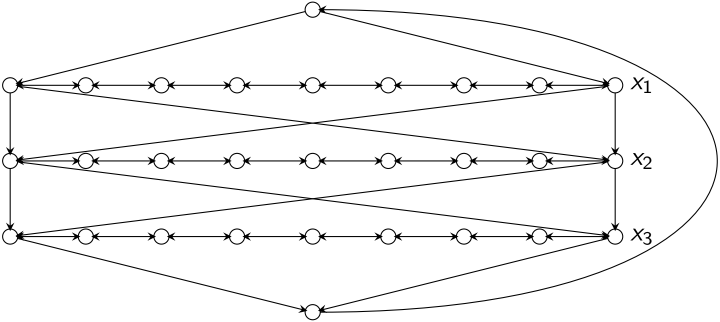
- Construct graph with 2^n Hamiltonian cycles, where each cycle corresponds to some boolean assignment
- Then add more graph structure to encode constraints on assignments imposed by the clauses

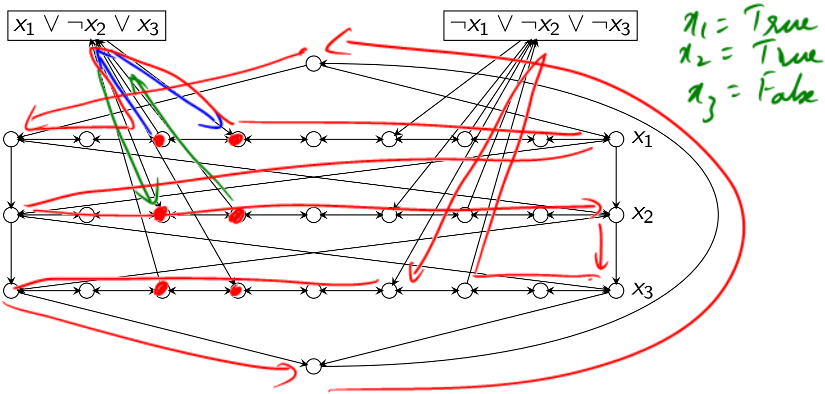
The Reduction: Phase I

- Traverse path i from left to right iff x_i is set to true
- Each path has $3(m+1)$ nodes where m is number of clauses in φ; nodes numbered from left to right (1 to $3m+3$)



The Reduction: Phase II

- Add vertex c_j for clause C_j. c_j has edge *from* vertex $3j$ and *to* vertex $3j + 1$ on path i if x_i appears in clause C_j, and has edge *from* vertex $3j + 1$ and *to* vertex $3j$ if $\neg x_i$ appears in C_j.



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$\Rightarrow$ Let a be the satisfying assignment for φ. Define Hamiltonian cycle as follows

- If $a(x_i) = 1$ then traverse path i from left to right
- If $a(x_i) = 0$ then traverse path i from right to left
- For each clause, path of at least one variable is in the "right" direction to splice in the node corresponding to clause □

Hamiltonian Cycle $\Rightarrow$ Satisfying assignment

Suppose Π is a Hamiltonian cycle in G_φ

- If Π enters c_j (vertex for clause C_j) from vertex $3j$ on path i then it must leave the clause vertex on edge to $3j+1$ on the *same path* i

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  - Thus, we don't have two unvisited neighbors (one to enter from, and the other to leave) to have a Hamiltonian Cycle

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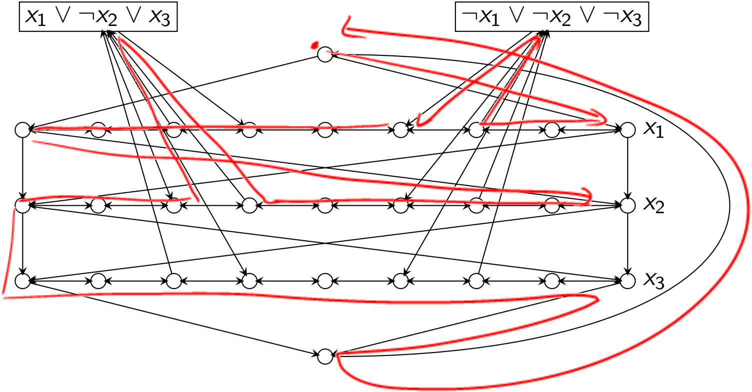
- Similarly, if Π enters c_j from vertex $3j+1$ on path i then it must leave the clause vertex c_j on edge to $3j$ on path i

Example



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- Thus, vertices visited immediately before and after C_i are connected by an edge
- We can remove c_j from cycle, and get Hamiltonian cyle in $G - c_j$
- Consider hamiltonian cycle in $G - \{c_1, \ldots c_m\}$; it traverses each path in only one direction, which determines the truth assignment

Hamiltonian Cycle

Problem

Input *Given undirected graph* $G = (V, E)$

Goal *Does* G *have a Hamiltonian cycle? That is, is there a cycle that visits every vertex exactly one (except start and end vertex)?*

NP-completeness

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Hamiltonian cycle problem for undirected graphs is NP-complete

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- The problem is in *NP*; proof left as exercise
- Hardness proved by reducing Directed Hamiltonian Cycle to this problem □

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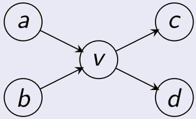
Goal: Given directed graph G, need to construct undirected graph G' such that G has Hamiltonian Path iff G' has Hamiltonian path

Reduction



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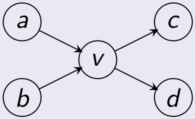

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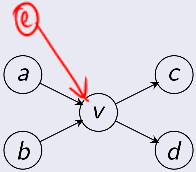

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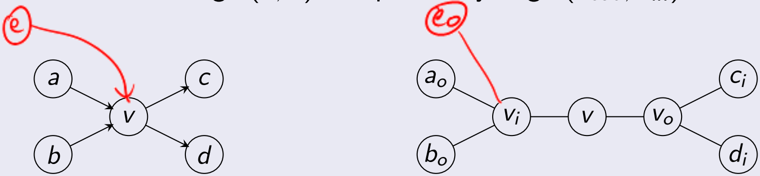

Reduction: Wrapup

- The reduction is polynomial time (exercise)
- The reduction is correct (exercise)

Graph Coloring

Input Given an undirected graph $G = (V, E)$ and integer k

Goal Can the vertices of the graph be colored using k colors so that vertices connected by an edge do not get the same color?

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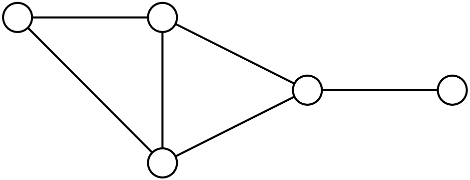

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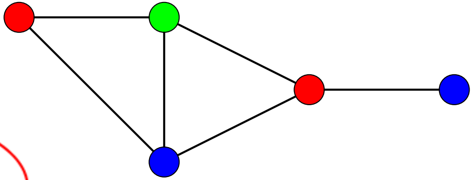

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G is 2-colorable iff G is bipartite! There is a linear time algorithm to check if G is bipartite using BFS (see Chapter 3 of Kleiberg-Tardos book).

Graph Coloring and Register Allocation

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Assign variables to (at most) k registers such that variables needed at the same time are not assigned to the same register

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Vertices are variables, and there is an edge between two vertices, if the two variables are “live” at the same time.

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- [Chaitin] Register allocation problem is equivalent to coloring the interference graph with k colors
- Moreover, 3-COLOR $\leq_P$ $k -$ Register Allocation, for any $k \geq 3$

Class Room Scheduling

Given n classes and their meeting times, are k rooms sufficient?

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Exercise: G is k-colorable iff k rooms are sufficient

Frequency Assignments in Cellular Networks

Cellular telephone systems that use Frequency Division Multiple Access (FDMA) (example: GSM in Europe and Asia and AT&T in USA)

- Breakup a frequency range $[a, b]$ into disjoint *bands* of frequencies $[a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]$
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Problem: given k bands and some region with n towers, is there a way to assign the bands to avoid interference?

Can reduce to k-coloring by creating intereference/conflict graph on towers

3-Coloring is NP-Complete

- 3-Coloring is in *NP*
  - Certificate: for each node a color from $\{1, 2, 3\}$
  - Certifier: Check if for each edge (u, v), the color of u is different from that of v
- Hardness: We will show 3-SAT $\leq_P$ 3-Coloring

Reduction Idea

Start with 3-SAT formula φ with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. Create graph G_φ such that G_φ is 3-colorable iff φ is satisfiable

- need to establish truth assignment for $x_1, \ldots, x_n$ via colors for some nodes in G_φ.

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- for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
- If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i

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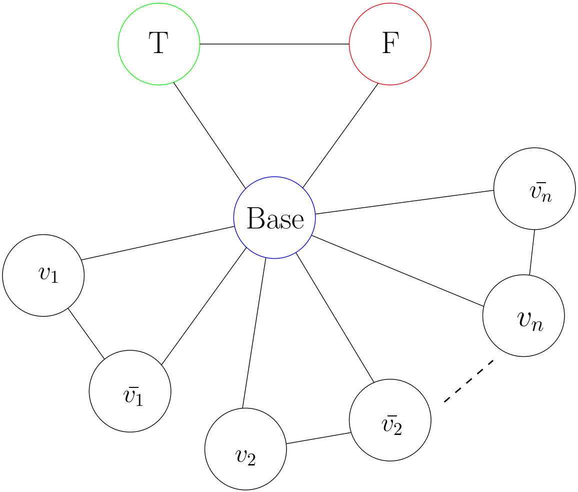
- Need to add constraints to ensure clauses are satisfied (next phase)

Figure



Clause Satisfiability Gadget

For each cluase $C_j = (a \vee b \vee c)$, create a small gadget graph

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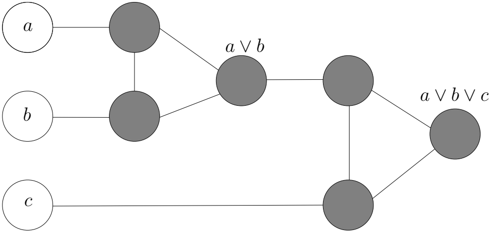
OR-gadget-graph:



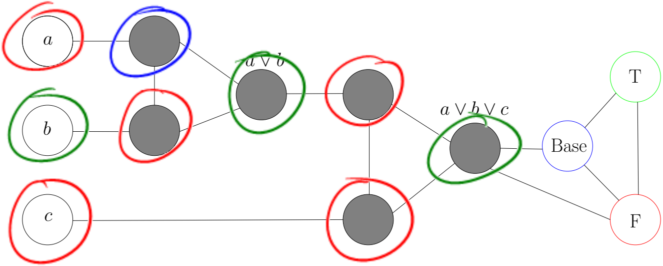
OR-Gadget Graph

Property: if a, b, c are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

Property: if one of a, b, c is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.

Reduction

- create triangle with nodes True, False, Base
- for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
- for each clause $C_j = (a \vee b \vee c)$, add OR-gadget graph with input nodes a, b, c and connect output node of gadget to both False and Base



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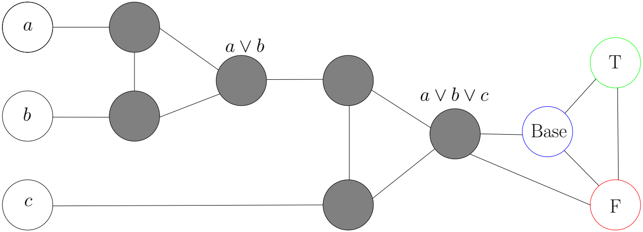


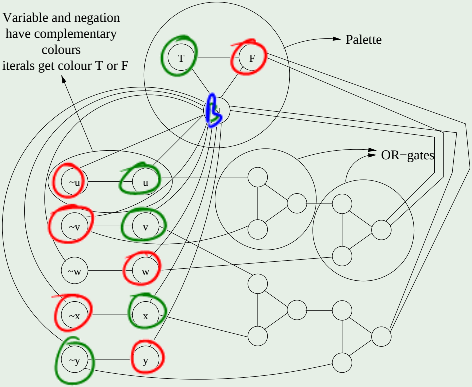
Claim

No legal 3-coloring of above graph (with coloring of nodes T, F, B fixed) in which a, b, c are colored False. If any of a, b, c are colored True then there is a legal 3-coloring of above graph.

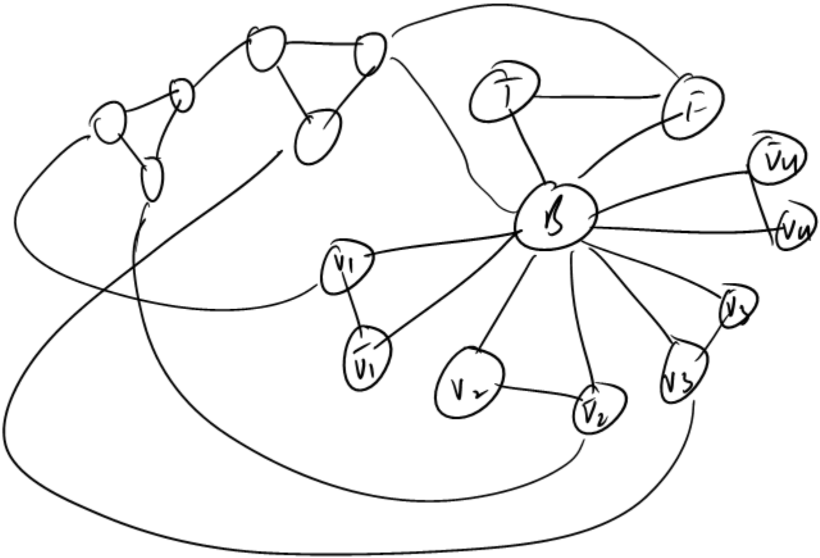
Reduction Outline

Example

$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$



$$\varphi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_4)$$



Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False

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G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment

Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

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- for each clause $C_j = (a \vee b \vee c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Other NP-Complete Problems

- 3-Dimensional Matching
- Subset Sum

Read book.

Need to Know NP-Complete Problems

- 3-SAT
- Circuit-SAT
- Independent Set
- Vertex Cover
- Clique
- Set Cover
- Hamiltonian Cycle in Directed/Undirected Graphs
- 3-Coloring
- 3-D Matching
- Subset Sum

Subset Sum and Knapsack

Subset Sum Problem: Given n integers $a_1, a_2, \ldots, a_n$ and a target B, is there a subset of S of $\{a_1, \ldots, a_n\}$ such that the numbers in S add up *precisely* to B?

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Subset Sum is NP-Complete — see book.

Subset Sum and Knapsack

Subset Sum Problem: Given n integers $a_1, a_2, \ldots, a_n$ and a target B, is there a subset of S of $\{a_1, \ldots, a_n\}$ such that the numbers in S add up *precisely* to B?

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Show Knapsack problem is NP-Complete via reduction from Subset Sum (exercise).

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Number problems of the above type are said to be *weakly* NP-Complete.