CS 473: Algorithms

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P and NP and Turing Machines

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- NP: set of decision problems that have polynomial time non-deterministic algorithms

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What is our model of computation?

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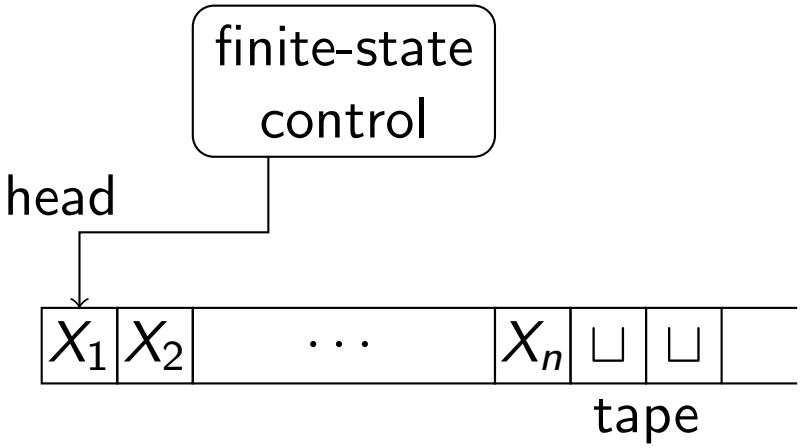
Question: What is an algorithm? Depends on the model of computation!

What is our model of computation?

Formally speaking our model of computation is Turing Machines.

Turing Machines: Recap



- Infinite tape
- Finite state control
- Input at beginning of tape
- Special tape letter "blank" $\sqcup$
- Head can move only one cell to left or right

Turing Machines: Formally

A TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$:

- Q is set of states in finite control
- q_0 start state, q_{accept} is accept state, q_{reject} is reject state
- Σ is input alphabet, Γ is tape alphabet (includes $\sqcup$)
- $\delta : Q \times \Gamma \rightarrow \{L, R\} \times \Gamma \times Q$ is transition function
  - $\delta(q, a) = (q', b, L)$ means that M in state q and head seeing a on tape will move to state q' while replacing a on tape with b and head moves left.

$L(M)$: language accepted by M is set of all input strings s on which M when started in q_0 on tape cell 1 and s on tape halts in q_{accept}.

P via TMs

Definition

M is a polynomial time TM if there is some polynomial $p(\cdot)$ such that on all inputs w, M halts in $p(|w|)$ steps.

Definition

L is a language in P iff there is a polynomial time TM M such that $L = L(M)$.

NP via TMs

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L is an NP language iff there is a *non-deterministic* polynomial time TM M such that $L = L(M)$.

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Non-deterministic TM: each step has a choice of moves

- $\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$.
  - Example: $\delta(q, a) = \{(q_1, b, L), (q_2, c, R), (q_3, a, R)\}$ means that M can non-deterministically choose one of the three possible moves from (q, a).
- $L(M)$: set of all strings s on which there *exists* some sequence of valid choices at each step that lead from q_0 to q_{accept}

Non-deterministic TMs vs Certifiers

Two definition of NP:

- L is in NP iff L has a polynomial time certifier $C(\cdot, \cdot)$.
- L is in NP iff L is decided by a non-deterministic polynomial time TM M.

Claim: Two definitions are equivalent. Why?

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Claim: Two definitions are equivalent. Why?

Informal proof idea: the certificate t for C corresponds to non-deterministic choices of M and vice-versa.
In other words L is in NP iff L is accepted by a NTM which first guesses a proof t of length poly in input $|s|$ and then acts as a *deterministic* TM.

Non-determinism, guessing and verification

- A non-deterministic machine has choices at each step and accepts a string if there *exists* a set of choices which lead to a final state.
- Equivalently the choices can be thought of as *guessing* a solution and then *verifying* that solution. In this view all the choices are made apriori and hence the verification can be deterministic. The "guess" is the "proof" and the "verifier" is the "certifier".
- We reemphasize the asymmetry inherent in the definition of non-determinism. Strings in the language can be easily verified. No easy way to verify that a string is not in the language.

Algorithms: TMs vs RAM Model

Why do we use TMs some times and RAM Model other times?

- TMs are very simple: no complicated instruction set, no jumps/pointers, no explicit loops etc.
  - Simplicity is useful in proofs
  - The "right" formal bare-bones model when dealing with subtelties
- RAM model is a closer approximation to the running time/space usage of realistic computers for reasonable problem sizes
  - Not appropriate for certain kinds of formal proofs when algorithms can take super-polynomial time and space

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Question

What is the hardest problem in *NP*? How do we define it?

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What is the hardest problem in *NP*? How do we define it?

Towards a definition

- Hardest problem must be in *NP*
- Hardest problem must be at least as "difficult" as every other problem in *NP*

NP-Complete Problems

Definition

A problem X is said to be *NP*-complete if

- $X \in NP$
- (Hardness) For any $Y \in NP$, $Y \leq_P X$

Solving *NP*-Complete Problems

Proposition

Suppose X is NP-complete. Then X can be solved in polynomial time iff $P = NP$

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- Thus, every problem $Y \in NP$ is such that $Y \in P$; $NP \subseteq P$
- Since $P \subseteq NP$, we have $P = NP$

$\Leftarrow$ Since $P = NP$, and $X \in NP$, we have a polynomial time algorithm for X $\square$

NP-Hard Problems

Definition

A problem X is said to be *NP*-hard if

- (Hardness) For any $Y \in NP$, $Y \leq_P X$

An *NP*-hard problem need not be in NP!

Example: Halting problem is NP-hard (why?) but not NP-complete.

Consequences of proving NP-completeness

If X is NP-complete

- Since we believe $P \neq NP$,

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X is unlikely to be efficiently solvable

At the very least, many smart people before you have failed to find an efficient algorithm for X

NP-Complete Problems

Question

Are there any problems that are *NP*-complete?

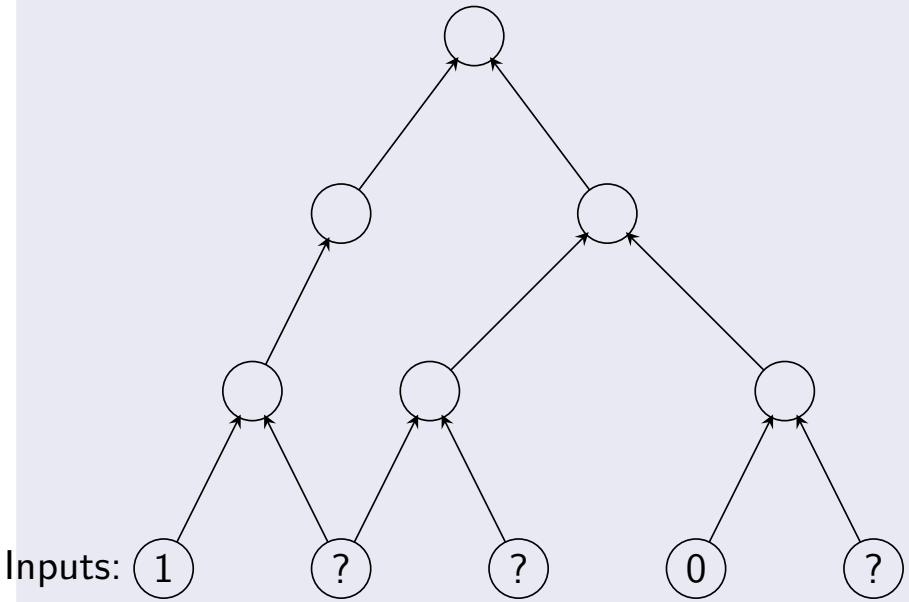
Answer

Yes! Many, many problems are *NP*-complete

Circuits

Definition

A circuit is a directed *acyclic* graph with



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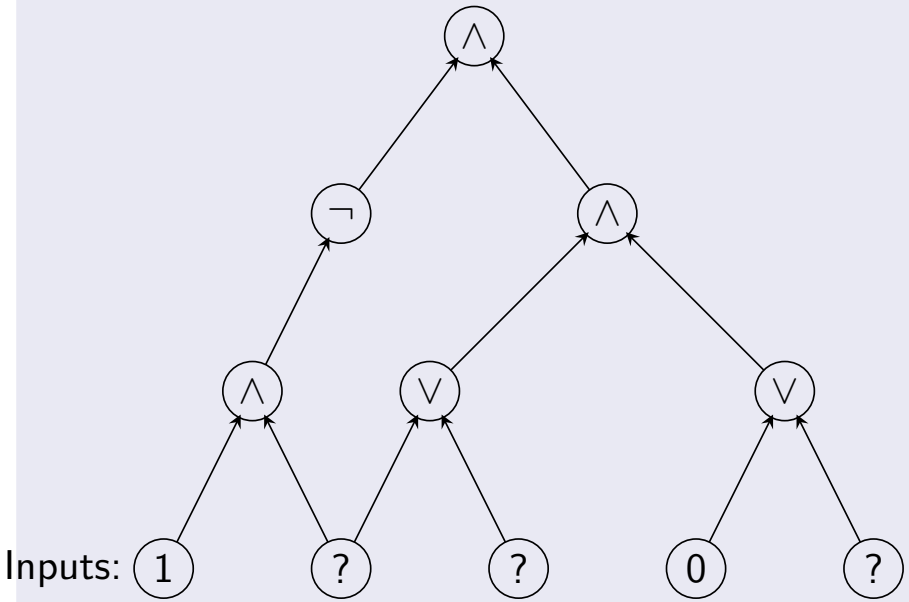

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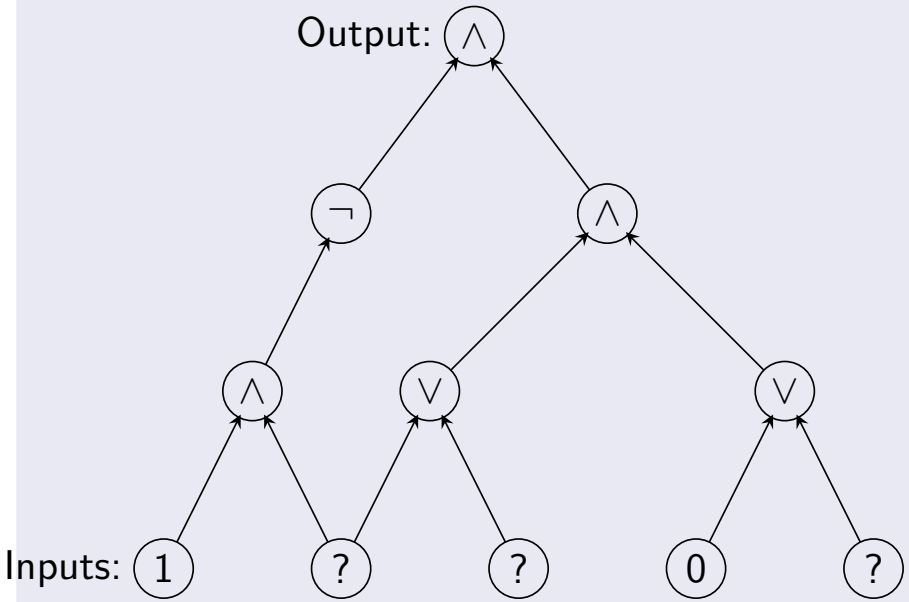

- Input vertices (without incoming edges) labelled with 0, 1 or a distinct variable
- Every other vertex is labelled $\vee$, $\wedge$ or $\neg$
- Single node output vertex with no outgoing edges

Cook-Levin Theorem

Circuit Satisfaction

Given a circuit as input, is there an assignment to the input variables that causes the output to get value 1?

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Circuit Satisfaction is NP-complete

Need to show

- Circuit Satisfaction is in NP
- *every* NP problem X reduces to Circuit Satisfaction

Circuit Satisfaction

Circuit Satisfaction is in *NP*.

- Certificate:
- Certifier:

Circuit Satisfaction

Circuit Satisfaction is in *NP*.

- Certificate: assignment to input variables
- Certifier: evaluate the value of each gate in a topological sort of DAG and check the output gate value

Circuit Satisfaction is NP-hard: Idea

Need to show that *every* NP problem X reduces to Circuit-SAT.

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$X \in NP$ implies that there are polynomials $p()$ and $q()$ and certifier/verfier program C such that for every string s the following is true:

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What does it mean that $X \in NP$?

$X \in NP$ implies that there are polynomials $p()$ and $q()$ and certifier/verfier program C such that for every string s the following is true:

- If s is a YES instance ($s \in X$) then there is a *proof* t of length $p(|s|)$ such that $C(s, t)$ says YES.
- If s is a NO instance ($s \notin X$) then for every string t of length at $p(|s|)$, $C(s, t)$ says NO.
- $C(s, t)$ runs in time $q(|s| + |t|)$ time (hence polynomial time)

Reducing X to Circuit-SAT

X is in NP means we have access to $p(), q(), C(\cdot, \cdot)$.

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What is $C(\cdot, \cdot)$? It is a program or equivalently a Turing Machine!
How are $p()$ and $q()$ given? As numbers.
Example: if 3 is given then $p() = n^3$.

Thus an NP problem is essentially a three tuple $< p, q, C >$ where C is either a program or TM.

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Same as the following: is there a proof t of length $p(|s|)$ such that $C(s, t)$ says YES.

How do we reduce X to Circuit-SAT?

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- We know that $|t| = p(|s|)$ so express boolean string t as $p(|s|)$ variables $t_1, t_2, \ldots, t_k$ where $k = p(|s|)$.
- Asking if there is a proof t that makes $C(s, t)$ say YES is same as whether there is an assignment of values to "unknown" variables $t_1, t_2, \ldots, t_k$ that will make G evaluate to true/YES.

Example: Independent Set

- Problem: Does $G = (V, E)$ have an independent set of size $\geq k$?
  - Certificate: Set $S \subseteq V$
  - Certifier: Check $|S| \geq k$ and no pair of vertices in S is connected by an edge

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Formally why is Independent Set in NP?

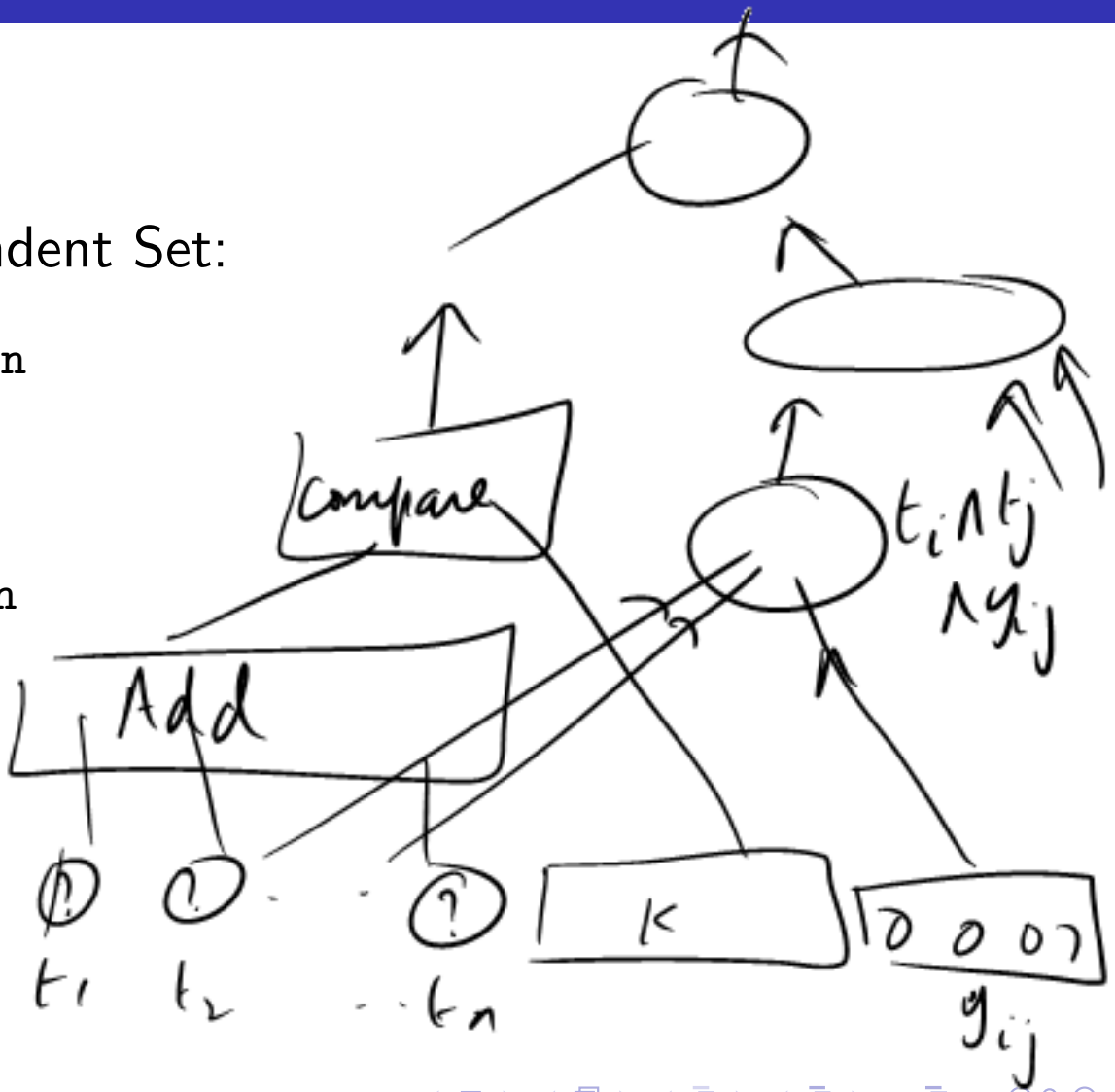
- Input:
  $< n, y_{1,1}, y_{1,2}, \ldots, y_{1,n}, y_{2,1}, \ldots, y_{2,n}, \ldots, y_{n,1}, \ldots, y_{n,n}, k >$ encodes $< G, k >$.
  - n is number of vertices in G
  - $y_{i,j}$ is a bit which is 1 if edge (i, j) is in G and 0 otherwise (adjacency matrix representation)
  - k is size of independent set
- Certificate: $t = t_1 t_2 \ldots t_n$. Interpretation is that t_i is 1 if vertex i is in the independent set, 0 otherwise.

Certifier for Independent Set

Certifier $C(s, t)$ for Independent Set:

```
if (t_1 + t_2 + ... + t_n < k) then
    return NO
else
    for each (i,j) do
        if (t_i ∧ t_j ∧ y_{i,j}) then
            return NO

return YES
```



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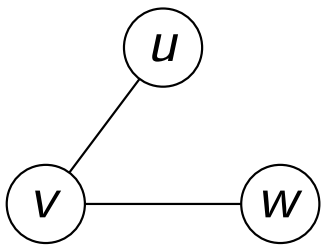
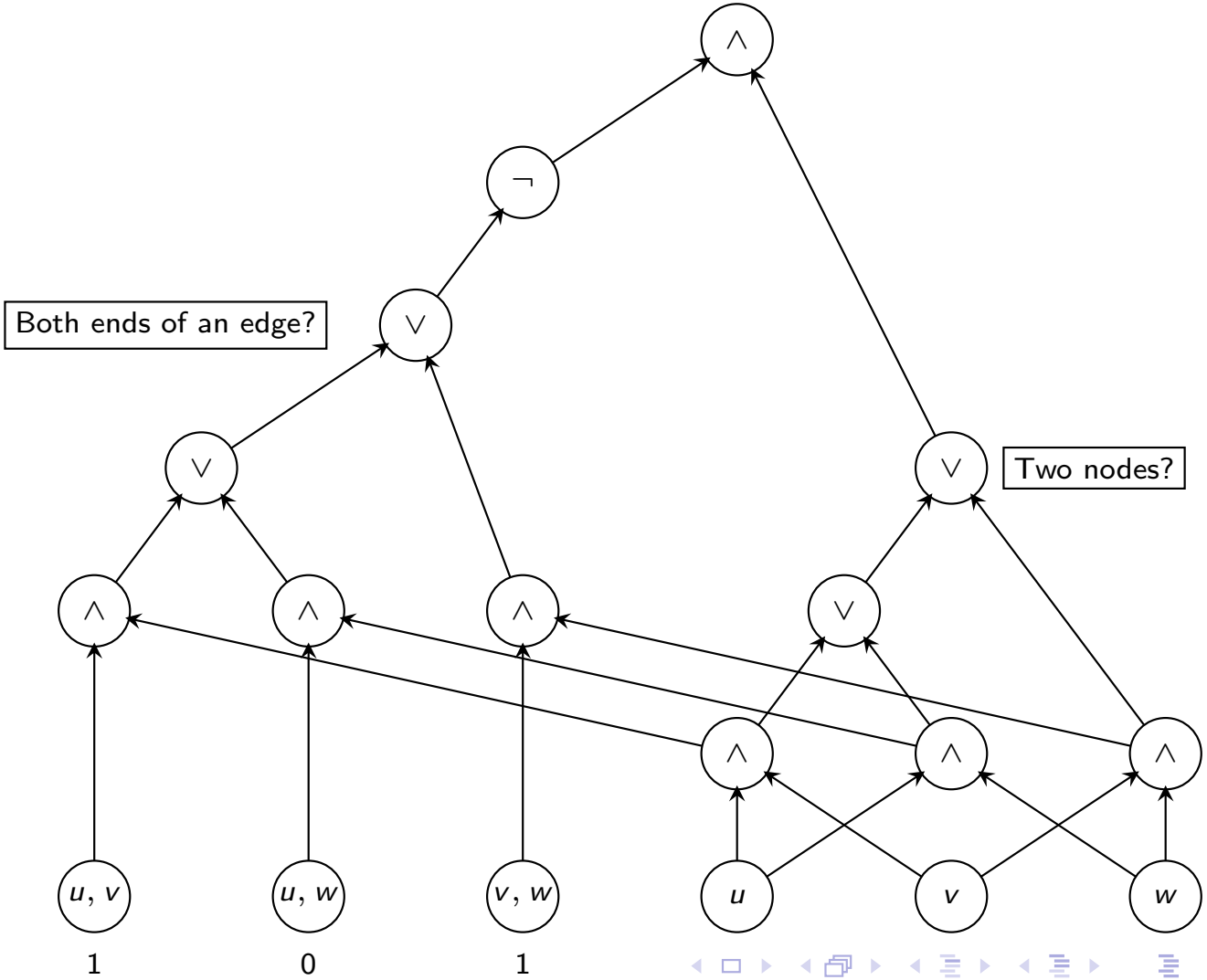


Figure: Graph G with $k = 2$



Circuit from Certifier

Programs, Turing Machines and Circuits

Consider "program" A that takes $f(|s|)$ steps on input string s.

Question: What computer is the program running on and what does *step* mean?

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Real computers difficult to reason with mathematically because

- instruction set is too rich
- pointers and control flow jumps in one step
- assumption that pointer to code fits in one word

Turing Machines

- simpler model of computation to reason with
- can simulate real computers with *polynomial* slow down
- all moves are *local* (head moves only one cell)

Certifiers that at TMs

Assume $C(\cdot, \cdot)$ is a (deterministic) Turing Machine M

Problem: Given M, input s, p, q decide if there is a proof t of length $p(|s|)$ such that M on s, t will halt in $q(|s|)$ time and say YES.

There is an algorithm $\mathcal{A}$ that can reduce above problem to Circuit-SAT mechanically as follows.

- $\mathcal{A}$ first computes $p(|s|)$ and $q(|s|)$.
- Knows that M can use at most $q(|s|)$ memory/tape cells
- Knows that M can run for at most $q(|s|)$ time
- Simulates the evolution of the state of M and memory over time using a big circuit

Simulation of Computation via Circuit

- Think of M's state at time ℓ as a string $x^\ell = x_1 x_2 \ldots x_k$ where each $x_i \in \{0, 1, B\} \times Q \cup \{q_{-1}\}$.
- At time 0 the state of M consists of input string s a guess t (unknown variables) of length $p(|s|)$ and rest $q(|s|)$ blank symbols.
- At time $q(|s|)$ we wish to know if M stops in q_{accept} with say all blanks on the tape.
- We write a circuit C_ℓ which captures the transistion of M from time ℓ to time $\ell + 1$.
- Composition of the circuits for all times 0 to $q(|s|)$ gives a big (still poly) sized circuit $\mathcal{C}$
- The final output of $\mathcal{C}$ should be true if and only if the entire state of M at the end leads to an accept state.

NP-hardness of Circuit Satisfaction

Key Ideas in reduction:

- Use TMs as the code for certifier for simplicity
- Since $p()$ and $q()$ are known to $\mathcal{A}$, it can set up all required memory and time steps in advance
- Simulate computation of the TM from one time to the next as a circuit that only looks at three adjacent cells at a time

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Note: Above reduction can be done to SAT as well. Reduction to SAT was the original proof of Steve Cook.

SAT is *NP*-complete

- We have seen that SAT $\in$ *NP*
- To show *NP*-hardness, we will reduce Circuit Satisfiability (CIR-SAT) to SAT

CIR-SAT $\leq_P$ SAT

Reduction

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- Case $\vee$: So $x_v = x_u \vee x_w$.

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Reduction

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- Case $\vee$: So $x_v = x_u \vee x_w$. In SAT, add clauses $(x_v \vee \neg x_u)$, $(x_v \vee \neg x_w)$, and $(\neg x_v \vee x_u \vee x_w)$

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- Case $\vee$: So $x_v = x_u \vee x_w$. In SAT, add clauses $(x_v \vee \neg x_u)$, $(x_v \vee \neg x_w)$, and $(\neg x_v \vee x_u \vee x_w)$
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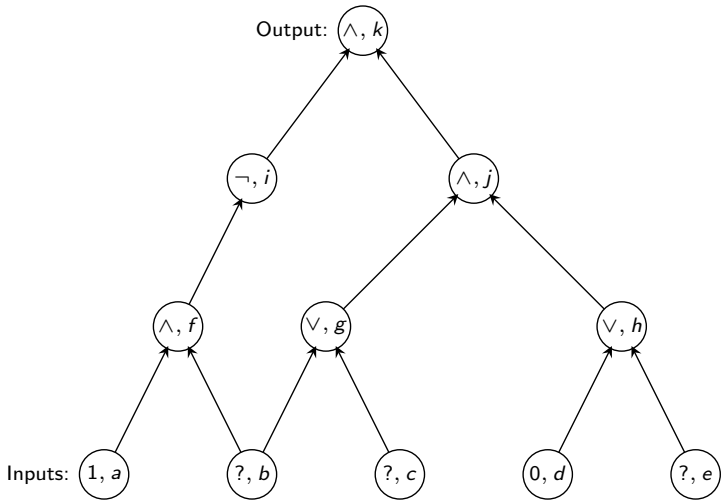
- Add the cluase x_v where v is the variable for the output gate

Example



SAT formula from circuit
Variables $x_a, x_b, \ldots x_k$, one for each gate
Relationship between variables given by gates and their input/outputs.

$$x_a = 1 \qquad x_d = 0$$
$$x_f = x_a \wedge x_b \qquad x_g = x_b \vee x_c \qquad x_h = x_d \vee x_e$$
$$x_i = \neg x_f \qquad x_j = x_g \wedge x_h$$
$$x_k = x_i \wedge x_j$$

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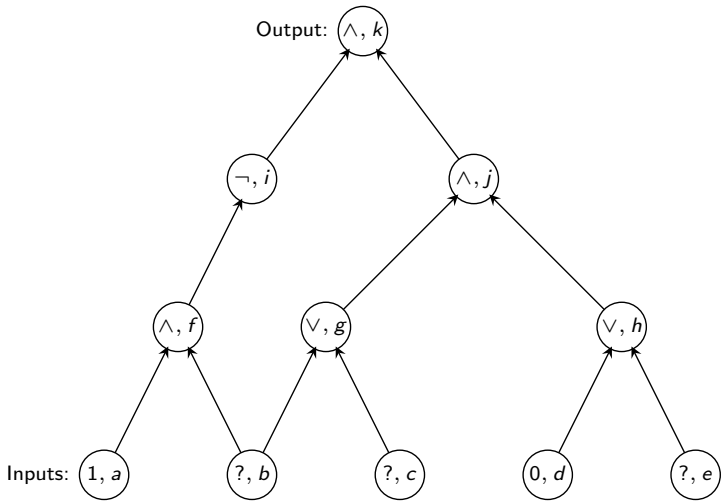

Express equalities using CNF clauses

x_a	$\neg x_d$	
$(\neg x_f \vee x_a)$	$(\neg x_f \vee x_b)$	$(x_f \vee \neg x_a \vee \neg x_b)$
$(x_g \vee \neg x_b)$	$(x_g \vee \neg x_c)$	$(\neg x_g \vee x_b \vee x_c)$
$(x_h \vee \neg x_d)$	$(x_h \vee \neg x_e)$	$(\neg x_h \vee x_d \vee x_e)$
$(x_i \vee x_f)$	$(\neg x_i \vee x_f)$	
$(\neg x_j \vee x_g)$	$(\neg x_j \vee x_h)$	$(x_j \vee \neg x_g \vee \neg x_h)$
$(\neg x_k \vee x_i)$	$(\neg x_k \vee x_j)$	$(x_k \vee \neg x_i \vee \neg x_j)$

Final SAT formula φ_C: conjunction of all of above cluases and the clause x_k, the output of the final gate

$$x_f = x_a \wedge x_b$$

$$\equiv (\neg x_f \Rightarrow (x_a \wedge x_b)) \wedge ((x_a \wedge x_b) \Rightarrow x_f)$$

$$\equiv (\neg x_f \vee (x_a \wedge x_b)) \wedge (\neg (x_a \wedge x_b) \vee x_f)$$

$$\equiv (\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \wedge (\neg x_a \vee \neg x_b \vee x_f)$$

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- Thus, a' satisfies C

Proving that a problem X is NP-complete

To prove X is NP-complete, show

- Show X is in NP.
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Transitivity of reductions:
$Y \leq_P SAT$ and $SAT \leq_P X$ and hence $Y \leq_P X$.

NP-Completeness via Reductions

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Hundreds and thousands of different problems from many areas of science and engineering have been shown to be NP-complete.

A surprisingly frequent phenomenon!

$P \neq NP$