

CS 473: Algorithms

Chandra Chekuri
chekuri@cs.illinois.edu
3228 Siebel Center

University of Illinois, Urbana-Champaign

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Part I

Reductions Continued

Polynomial Time Reduction

A polynomial time reduction from a *decision* problem X to a *decision* problem Y is an *algorithm* \mathcal{A} that has the following properties:

- given an instance I_X of X , \mathcal{A} produces an instance I_Y of Y
- \mathcal{A} runs in time polynomial in $|I_X|$. This implies that $|I_Y|$ (size of I_Y) is polynomial in $|I_X|$
- Answer to I_X YES *iff* answer to I_Y is YES.

Notation: $X \leq_P Y$ if X reduces to Y

Proposition

If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X .

Such a reduction is called a Karp reduction. Most reductions we will need are Karp reductions

A More General Reduction

Turing Reduction (the one given in the book)

Problem X polynomial time reduces to Y if there is an algorithm \mathcal{A} for X that has the following properties:

- on any given instance I_X of X , \mathcal{A} uses polynomial in $|I_X|$ “steps”
- a step is either a standard computation step or
- a sub-routine call to an algorithm that solves Y

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- a sub-routine call to an algorithm that solves Y

Note: In making sub-routine call to algorithm to solve Y , \mathcal{A} can only ask questions of size polynomial in $|I_X|$. Why?

Above reduction is called a Turing reduction.

Example of Turing Reduction

Input Collection of arcs on a circle.

Goal Compute the maximum number of non-overlapping arcs.

Reduced to the following problem:?

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How?

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How? Used algorithm for interval problem multiple times.

Turing vs Karp Reductions

- Turing reductions more general than Karp reductions
- Turing reduction useful in obtaining algorithms via reductions
- Karp reduction is simpler and easier to use to prove hardness of problems
- Perhaps surprisingly, Karp reductions, although limited, suffice for most known NP-Completeness proofs

Propositional Formulas

Definition

Consider a set of boolean variables x_1, x_2, \dots, x_n

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- A formula φ is in 3CNF if it is a CNF formula such that every clause has exactly 3 literals
 - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$ is a 3CNF formula, but $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is not.

Satisfiability

SAT

Given a CNF formula φ , is there a truth assignment to variables such that φ evaluates to true?

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Example

$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is satisfiable; take x_1, x_2, \dots, x_5 to be all true

$(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee x_2)$ is not satisfiable

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3-SAT

Given a 3-CNF formula φ , is there a truth assignment to variables such that φ evaluates to true?

Importance of SAT and 3-SAT

- SAT and 3-SAT are basic constraint satisfaction problems
- Many different problems can be reduced to them because of the simple yet powerful expressivity of logical constraints
- Arise naturally in many applications involving hardware and software verification and correctness
- As we will see, it is a fundamental problem in theory of NP-Completeness

$\text{SAT} \leq_P \text{3-SAT}$

Easy to see that $\text{3-SAT} \leq_P \text{SAT}$. A 3-SAT instance is also an instance of SAT.

We can show that $\text{SAT} \leq_P \text{3-SAT}$.

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Given φ a SAT formula we create a 3-SAT formula φ' such that

- φ is satisfiable iff φ' is satisfiable
- φ' can be constructed from φ in time polynomial in $|\varphi|$.

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- φ is satisfiable iff φ' is satisfiable
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Idea: if a clause of φ is not of length 3, replace it with several clauses of length exactly 3

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Reduction Ideas

Challenge: Some of the clauses in φ may have less or more than 3 literals. For each clause with < 3 or > 3 literals, we will construct a set of logically equivalent clauses.

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Reduction Ideas

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- **Case clause with 1 literal:** Let $c = \ell$. Let u, v be new variables. Consider

$$c' = (\ell \vee u \vee v) \wedge (\ell \vee u \vee \neg v) \wedge (\ell \vee \neg u \vee v) \wedge (\ell \vee \neg u \vee \neg v)$$

Observe that c' is satisfiable iff c is satisfiable

$$\phi = x_1 \quad (y \quad z)$$

$$\phi' = (x_1, v y, v z) \wedge (x_1, v y, v \bar{z})$$

$$\wedge (x_1, v \bar{y}, v z) \wedge (x_1, v \bar{y}, v \bar{z})$$

$$\phi = x_1 \vee x_2$$

$$y_1, z_1, y_2, z_2$$

$$\phi' = (x$$

SAT \leq_P 3-SAT (contd)

Reduction Ideas: 2 and more literals

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- **Case clause with 2 literals:** Let $c = l_1 \vee l_2$. Let u be a new variable. Consider

$$c' = (l_1 \vee l_2 \vee u) \wedge (l_1 \vee l_2 \vee \neg u)$$

Again c is satisfiable iff c' is satisfiable

SAT \leq_P 3-SAT (contd)

Reduction Ideas: 2 and more literals

- **Case clause with > 3 literals:** Let $c = l_1 \vee \dots \vee l_k$. Let u_1, \dots, u_{k-3} be new variables. Consider

$$c' = (l_1 \vee l_2 \vee u_1) \wedge (l_3 \vee \neg u_1 \vee u_2) \wedge (l_4 \vee \neg u_2 \vee u_3) \wedge \dots \wedge (l_{k-2} \vee \neg u_{k-4} \vee u_{k-3}) \wedge (l_{k-1} \vee l_k \vee \neg u_{k-3})$$

c is satisfiable iff c' is satisfiable

Another way to see it — reduce size of clause by one:

$$c' = (l_1 \vee l_2 \dots \vee l_{k-2} \vee u_{k-3}) \wedge (l_{k-1} \vee l_k \vee \neg u_{k-3})$$

$$\phi = (x_1 \vee x_2 \vee \bar{x}_3 \vee x_4)$$

$$\phi' = (x_1 \vee x_2 \vee z) (\bar{x}_3 \vee x_4 \vee \bar{z})$$

$$\phi = \underline{(x_1 \vee x_2 \vee \bar{x}_3 \vee x_4 \vee \bar{x}_5)}$$

$$\phi' = (x_1 \vee x_2 \vee \bar{x}_3 \vee z) (x_4 \vee \bar{x}_5 \vee \bar{z})$$

An Example

Example

$$\varphi = (\neg x_1 \vee \neg x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1) \wedge (x_1).$$

$$\psi = (\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z)$$

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Overall Reduction Algorithm

```
Input: CNF formula  $\varphi$ 
for each clause  $c$  of  $\varphi$ 
    if  $c$  does not have exactly 3 literals
        construct  $c'$  as before
    else
         $c' = c$ 
 $\psi$  is conjunction of all  $c'$  constructed in loop
is  $\psi$  satisfiable?
```

Correctness (informal)

φ is satisfiable iff ψ is satisfiable because for each clause c , the new 3-CNF formula c' is logically equivalent to c .

What about 2-SAT?

2-SAT can be solved in polynomial time!

No known polynomial time reduction from SAT (or 3-SAT) to 2-SAT. If there was, then SAT and 3-SAT would be solvable in polynomial time.

3-SAT \leq_P Independent Set

Input Given a 3-CNF formula φ

Goal Construct a graph G_φ and number k such that G_φ has an independent set of size k iff φ is satisfiable.

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Importance of reduction: Although 3-SAT is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.

Interpreting 3-SAT

There are two ways to think about 3-SAT

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- Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true
- Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in **conflict**, i.e., you pick x_i and $\neg x_i$

We will take the second view of 3-SAT to construct the reduction.

The Reduction

- G_φ will have one vertex for each literal in a clause

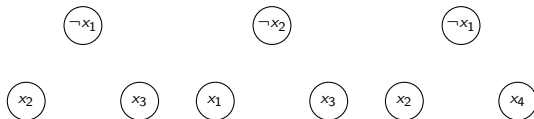


Figure: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

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- Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true

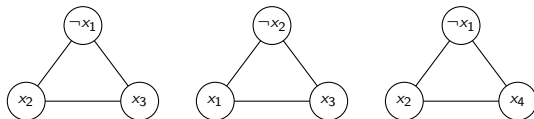


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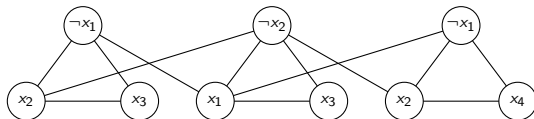


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- Take k to be the number of clauses

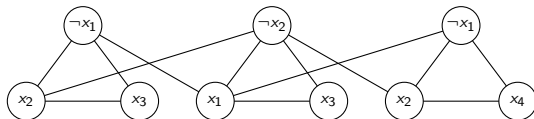


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$a_1 = 1$ $a_2 = 0$, $a_3 = 1$ $a_4 = 1$

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- \Rightarrow Let a be the truth assignment satisfying φ
- Pick one of the vertices, corresponding to true literals under a , from each triangle. This is an independent set of the appropriate size □

Correctness (contd)

Proposition

φ is satisfiable iff G_φ has an independent set of size k (= number of clauses in φ).

Proof.

\Leftarrow Let S be an independent set of size k

- S must contain exactly one vertex from each clause
- S cannot contain vertices labelled by conflicting clauses
- Thus, it is possible to obtain a truth assignment that makes in the literals in S true; such an assignment satisfies one literal in every clause □

Transitivity of Reductions

$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.

Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y
In other words show that an algorithm for Y implies an algorithm for X .

Part II

Definition of NP

Recap ...

Problems

- Independent Set
- Vertex Cover
- Set Cover
- SAT
- 3-SAT

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3-SAT \leq_P Independent Set $\stackrel{\leq_P}{\geq_P}$ Vertex Cover

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 $3\text{-SAT} \leq_P \text{SAT} \leq_P 3\text{-SAT}$

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Definition

- A is an **algorithm for problem** X if $A(s) = \text{“yes”}$ iff $s \in X$
- A is said to have a **polynomial running time** if there is a polynomial $p(\cdot)$ such that for every string s , $A(s)$ terminates in at most $O(p(|s|))$ steps

Polynomial Time

Definition

Polynomial time (denoted P) is the class of all (decision) problems that have an algorithm that solves it in polynomial time

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Example

Problems in P include

- Is there a shortest path from s to t of length $\leq k$ in G ?
- Is there a flow of value $\geq k$ in network G ?
- Is there an assignment to variables to satisfy given linear constraints?

Efficiency Hypothesis

A problem X has an efficient algorithm iff $X \in P$, that is X has a polynomial time algorithm.

Justifications:

- robustness of definition to variations in machines
- a sound theoretical definition
- most known polynomial time algorithms for “natural” problems have small polynomial running times

Problems with no known polynomial time algorithms

Problems

- Independent Set
- Vertex Cover
- Set Cover
- SAT
- 3-SAT

There are of course undecidable problems (no algorithm at all!) but many problems that we want to solve are like above.

Question: What is common to above problems?

Efficient Checkability

Above problems share the following feature:

For any YES instance I_X of X there is a proof/certificate/solution that is of length $\text{poly}(|I_X|)$ such that given a proof one can efficiently check that I_X is indeed a YES instance

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Examples:

- SAT formulat φ : proof is a satisfying assignment
- Independent Set in graph G and k : a subset S of vertices

Certifiers

Definition

An algorithm $C(\cdot, \cdot)$ is a **certifier** for problem X if for every $s \in X$ there is some string t such that $C(s, t) = \text{"yes"}$, and conversely, if for some s and t , $C(s, t) = \text{"yes"}$ then $s \in X$.

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The string t is called a **certificate** or **proof** for s

Efficient Certifier

C is an **efficient certifier** for problem X if there is a polynomial $p(\cdot)$ such that for every string s , $s \in X$ iff there is a string t with $|t| \leq p(|s|)$, $C(s, t) = \text{"yes"}$ and C runs in polynomial time

Example: Independent Set

- **Problem:** Does $G = (V, E)$ have an independent set of size $\geq k$?

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Example: Vertex Cover

- **Problem:** Does G have a vertex cover of size $\leq k$?

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Example: Vertex Cover

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Example: SAT

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Example: Composites

- **Problem:** Is number s a composite?

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 - **Certificate:** A factor $t \leq s$ such that $t \neq 1$ and $t \neq s$

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- **Problem:** Is number s a composite?
 - **Certificate:** A factor $t \leq s$ such that $t \neq 1$ and $t \neq s$
 - **Certifier:** Check that t divides s (Euclid's algorithm)

Nondeterministic Polynomial Time

Definition

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Example

Independent Set, Vertex Cover, Set Cover, SAT, 3-SAT, Composites are all examples of problems in *NP*

Asymmetry in Definition of NP

Note that only YES instances have a short proof/certificate. NO instances need not have a short certificate.

Example: SAT formula φ . No easy way to prove that φ is NOT satisfiable!

More on this and co-NP later on.

P versus NP

Proposition

$$P \subseteq NP$$

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Consider problem $X \in P$ with algorithm A . Need to demonstrate that X has an efficient certifier

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- Certifier C on input s, t , runs $A(s)$ and returns the answer
- C runs in polynomial time
- If $s \in X$ then for every t , $C(s, t) = \text{"yes"}$
- If $s \notin X$ then for every t , $C(s, t) = \text{"no"}$



Exponential Time

Definition

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Example: $O(2^n)$, $O(2^{n \log n})$, $O(2^{n^3})$, ...

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Let $X \in NP$ with certifier C . Need to design an exponential time algorithm for X

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- The above algorithm correctly solves X (exercise)
- Algorithm runs in $O(q(|s| + |p(s)|)2^{p(|s|)})$, where q is the running time of C



Examples

- SAT: try all possible truth assignment to variables
- Independent set: try all possible subsets of vertices
- Vertex cover: try all possible subsets of vertices

Is *NP* efficiently solvable?

We know $P \subseteq NP \subseteq EXP$

Is NP efficiently solvable?

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Big Question

Is there are problem in NP that **does not** belong to P ? Is $P = NP$?

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If $P = NP$...

- Many important optimization problems can be solved efficiently
- The RSA cryptosystem can be broken
- No security on the web
- No e-commerce ...
- Creativity can be automated! Proofs for mathematical statement can be found by computers automatically (if short ones exist)

P versus NP

Status

Relationship between P and NP remains one of the most important open problems in mathematics/computer science

Consensus: Most people feel $P \neq NP$

Resolving P versus NP is a Clay Millenium Prize Problem. You can win a million dollars in addition to a Turing award and major fame!

$$L = \{ w \in \{0,1\}^* \mid w \text{ ends in } 010010010 \}$$

