CS 473: Algorithms

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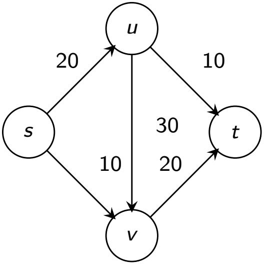
Fall 2010

Part I

Algorithm(s) for Maximum Flow

Greedy Approach



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3. *Augment* flow along this path
4. Repeat augmentation for as long as possible.

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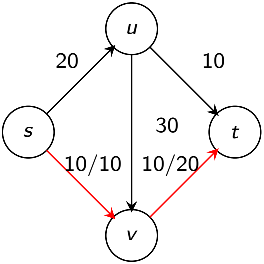

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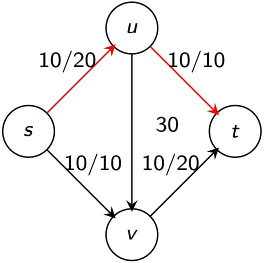

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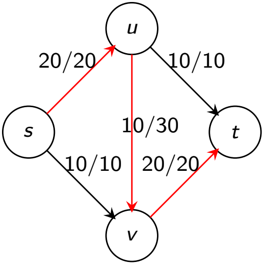

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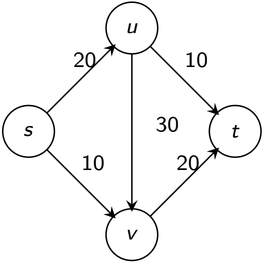

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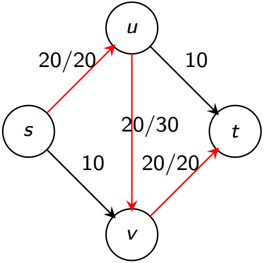

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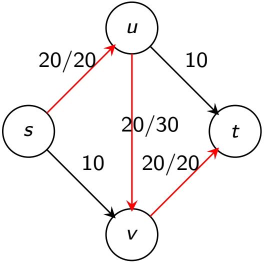

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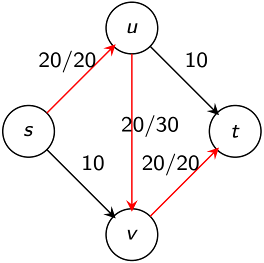

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Need to "push-back" flow along edge (u, v)

Residual Graph

Definition

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- Backward Edges: For each edge $e = (u, v) \in E$ with $f(e) > 0$, we $(v, u) \in E'$ with capacity $f(e)$

Residual Graph Example

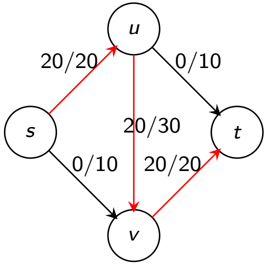

Figure: Flow in red edges

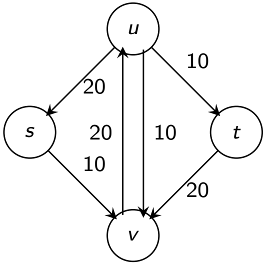

Figure: Residual Graph

Residual Graph Property

Observation: Residual graph captures the “residual” problem exactly.

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Lemma

Let f and f' be two flows in G with $v(f') \geq v(f)$. Then there is a flow f'' of value $v(f') - v(f)$ in G_f.

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Definition of + and - for flows is intuitive and the above lemmas are easy in some sense but a bit messy to formally prove.

Residual Graph Property: Implication

Recursive algorithm for finding a maximum flow:

```
MaxFlow(G, s, t):
    If the flow from s to t is 0
        return 0
    Find any flow f with v(f) > 0 in G
    Recursively compute a maximum flow f' in G_f
    Output the flow f + f'
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Iterative algorithm for finding a maximum flow:

```
MaxFlow(G, s, t):
    Start with flow f that is 0 on all edges
    While there is a flow f' in G_f with v(f') > 0 do
        f = f + f'
        Update G_f
    endWhile
    Output f
```

Ford-Fulkerson Algorithm

```
for every edge e, f(e) = 0
G_f is residual graph of G with respect to f
while G_f has a simple s-t path
    let P be simple s-t path in G_f
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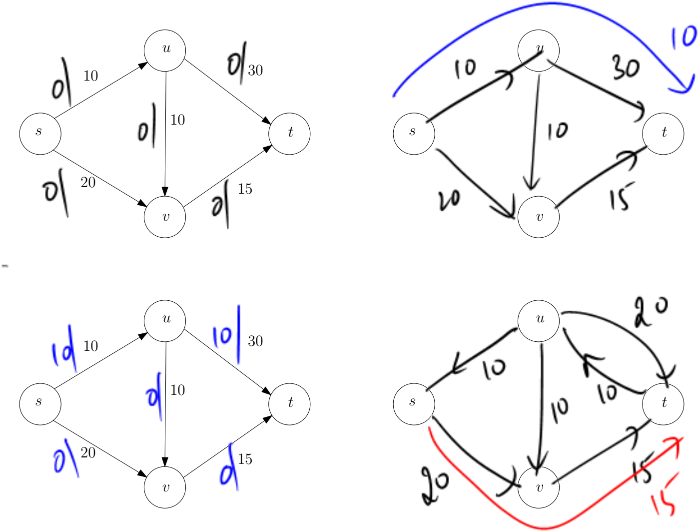
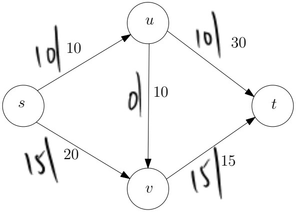
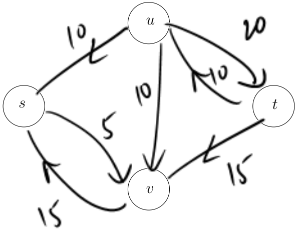
augment(f,P)
    let b be bottleneck capacity, i.e., min capacity of edges in P
    for each edge (u,v) in P
        if e=(u,v) is a forward edge
            f(e) = f(e) + b
        else (* (u,v) is a backward edge *)
            let e = (v,u) (* (v,u) is in G *)
            f(e) = f(e) - b
    return f
```

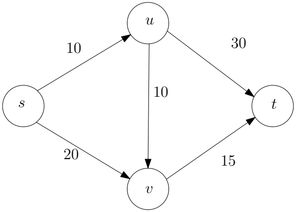
Example

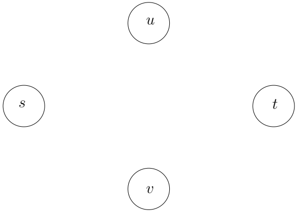


Example continued









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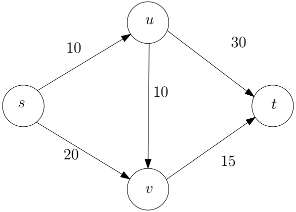

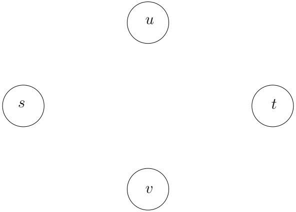

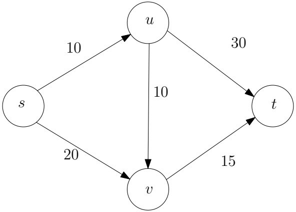

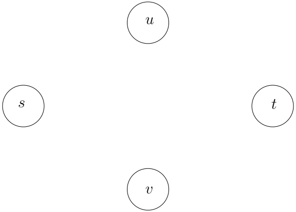

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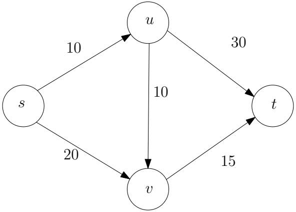

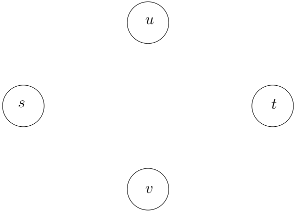

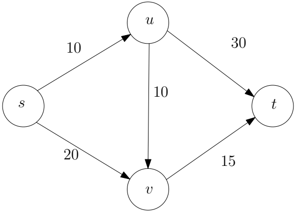

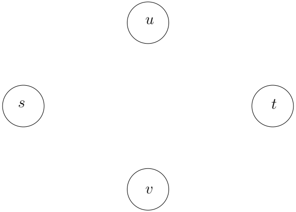

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Lemma

If f is a flow and P is a simple s-t path in G_f, then $f' = \texttt{augment}(f, P)$ is also a flow.

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- Conservation constraint: Let v be an internal node. Let e_1, e_2 be edges of P incident to v. Four cases based on whether e_1, e_2 are forward or backward edges. Check cases (see fig next slide). □

Properties about Augmentation: Conservation Constraint

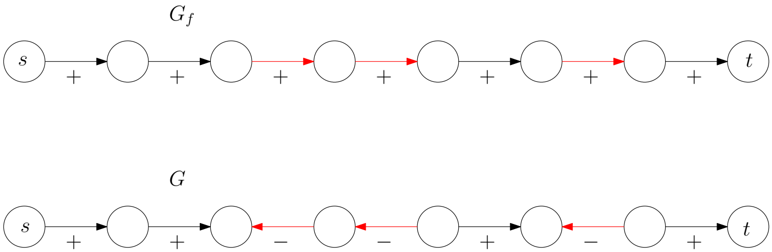

Figure: Augmenting path P in G_f and corresponding change of flow in G. Red edges are backward edges.

Properties about Augmentation: Integer Flow

Lemma

At every stage of the Ford-Fulkerson algorithm, the flow values $f(e)$ and the residual capacities in G_f are integers

Proof.

Initial flow and residual capacities are integers. Suppose lemma holds for j iterations. Then in $j+1$st iteration, minimum capacity edge b is an integer, and so flow after augmentation is an integer. □

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- P is simple and so never returns to s
- Thus, value of flow increases by the flow on edge e □

Termination Proof

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Let C be the minimum cut value; in particular $C \leq \sum_{e \text{ out of } s} c(e)$. Ford-Fulkerson algorithm terminates after finding at most C augmenting paths.

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- Running time is $O(C(n+m))$ (or $O(mC)$).

Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the running time be as $\Omega(mC)$ or is our analysis weak?

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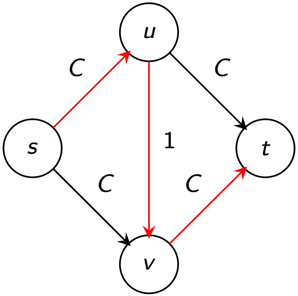

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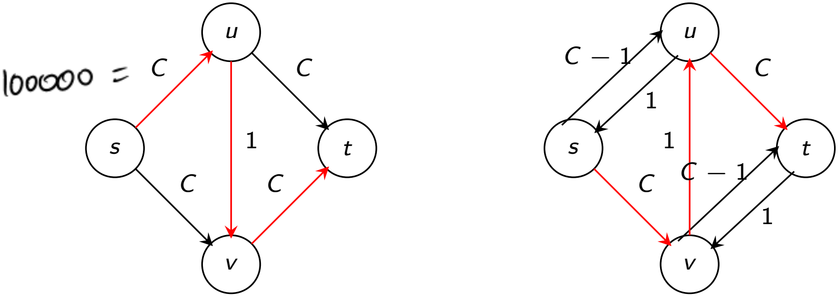

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Proof idea: show a cut of value equal to the flow. Also shows that maximum flow is equal to minimum cut!

Recalling Cuts

Definition

Given a flow network an s-t cut is a set of edges $E' \subset E$ such that removing E' *disconnects* s from t: in other words there is no directed $s \rightarrow t$ path in $E - E'$. *Capacity* of cut E' is $\sum_{e \in E'} c(e)$.

Let $A \subset V$ such that

- $s \in A$, $t \notin A$
- $B = V - A$ and hence $t \in B$

Define $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$

Claim

(A, B) *is an s-t cut.*

Recall: Every *minimal* s-t cut E' is a cut of the form (A, B).

Ford-Fulkerson Correctness

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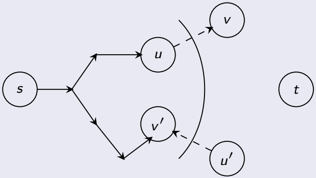

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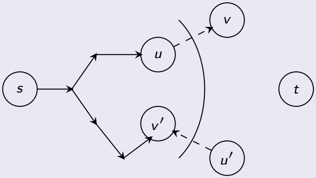
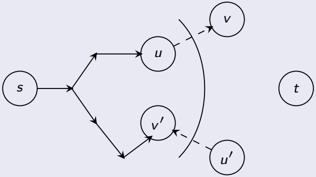


- $s \in A$ and $t \in B$. So (A, B) is an s-t cut in G
- If $e = (u, v) \in G$ with $u \in A$ and $v \in B$, then $f(e) = c(e)$ (saturated edge) because otherwise v is reachable from s in G_f

□

Lemma Proof Continued

Proof.

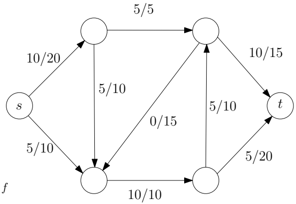


- If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s in G_f
- Thus,

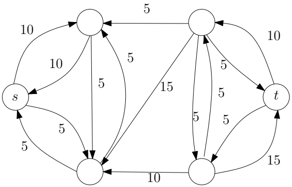
$$
\begin{aligned}
v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\
&= f^{\text{out}}(A) - 0 \\
&= c(A, B) - 0 \\
&= c(A, B)
\end{aligned}
$$

□

Example

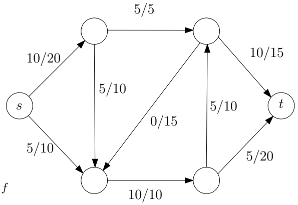


Flow f



Residual graph G_f: no s-t path

Example



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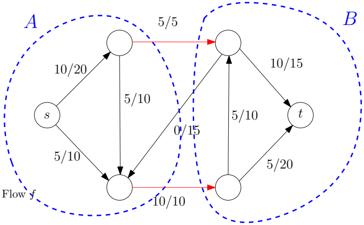

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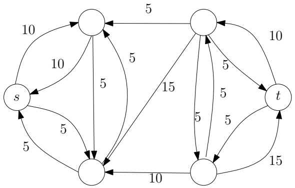

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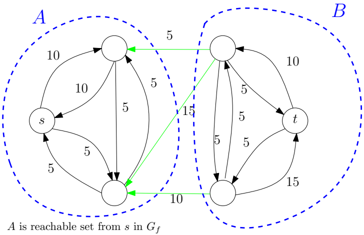

A is reachable set from s in G_f

Ford-Fulkerson Correctness

Theorem

The flow returned by the algorithm is the maximum flow.

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- For any flow f and s-t cut (A, B), $v(f) \leq c(A, B)$
- For flow f^* returned by algorithm, $v(f^*) = c(A^*, B^*)$ for some s-T cut (A^*, B^*)

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Proof.

- For any flow f and s-t cut (A, B), $v(f) \leq c(A, B)$
- For flow f^* returned by algorithm, $v(f^*) = c(A^*, B^*)$ for some s-T cut (A^*, B^*)
- Hence, f^* is maximum

□

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network G, the value of a maximum s-t flow is equal to the capacity of the minimum s-t cut.

Proof.

Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. □

Theorem

For any network G with integer capacities, there is a maximum s-t flow that is integer valued.

Proof.

Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. □

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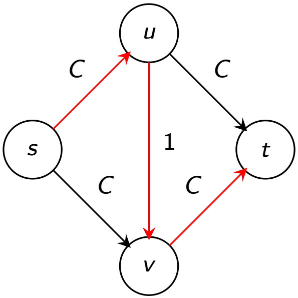

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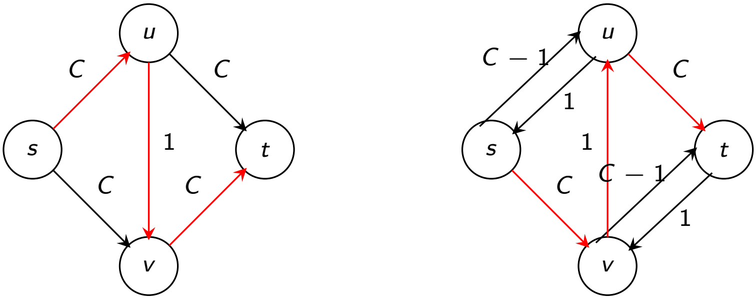

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Polynomial Time Algorithms

Question: Is there a polynomial time algorithm for maxflow?

Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way? Yes! Two variants.

- Choose the augmenting path with largest bottleneck capacity.
- Choose the shortest augmenting path.

Augmenting Paths with Large Bottleneck Capacity

- Pick augmenting paths with largest bottleneck capacity in each iteration of Ford-Fulkerson
- How do we find path with largest bottleneck capacity?

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- How do we find path with largest bottleneck capacity?
  - Assume we know Δ the bottleneck capacity
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  - Do binary search to find largest Δ
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Book gives a simpler variant called Capacity Scaling algorithm that runs in $O(m^2 \log C)$ time.

Augmenting Paths with Large Bottleneck Capacity

How do we find path with largest bottleneck capacity?

- Max bottleneck capacity is one of the edge capacities. Why?
- Can do binary search on the edge capacities. First, sort the edges by their capacities and then do binary search on that array as before.
- Algorithm's running time is $O(m \log m)$.
- Different algorithm that also leads to $O(m \log m)$ time algorithm by adapting Prim's algorithm.

Removing Dependence on C

- [Edmonds-Karp, Dinitz] Picking augmenting paths with fewest number of edges yields a $O(m^2 n)$ algorithm, i.e., independent of C. Such an algorithm is called a strongly polynomial time algorithm since the running time does not depend on the numbers (assuming RAM model). (Many implementation of Ford-Fulkerson would actually use shortest augmenting path if they use BFS to find an s-t path).

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- Further improvements can yield algorithms running in $O(mn \log n)$, or $O(n^3)$.

Finding a Minimum Cut

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Finding a Minimum Cut

Question: How do we find an actual minimum s-t cut?
Proof gives the algorithm!

- Compute an s-t maximum flow f in G
- Obtain the residual graph G_f
- Find the nodes A reachable from s in G_f
- Output the cut $(A, B) = \{(u, v) \mid u \in A, v \in B\}$. Note: The cut is found in G while A is found in G_f

Running time is essentially the same as finding a maximum flow.

Note: Given G and a flow f there is a linear time algorithm to check if f is a maximum flow and if it is, outputs a minimum cut. How?