CS 473: Algorithms

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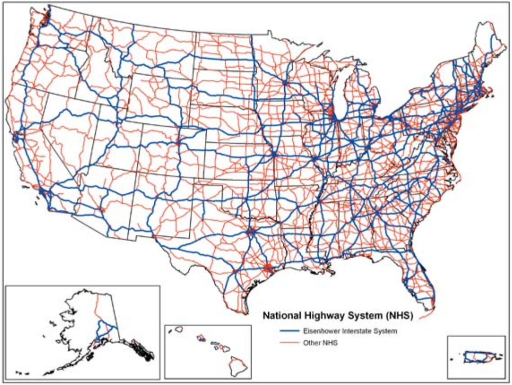
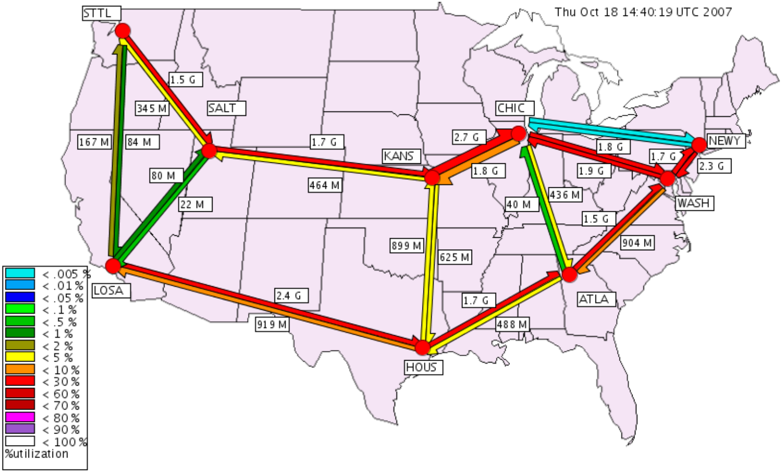
Part I

Network Flows: Introduction and Setup

Transportation/Road Network



Internet Backbone Network



Common Features of Flow Networks

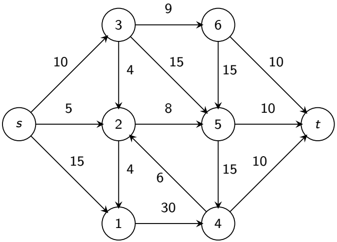
- *Network* represented by a (directed) *graph* $G = (V, E)$
- Each edge e has a *capacity* $c(e) \geq 0$ that limits amount of *traffic* on e
- *Source(s)* of traffic/data
- *Sink(s)* of traffic/data
- Traffic *flows* from sources to sinks
- Traffic is *switched/interchanged* at nodes

Flow: abstract term to indicate stuff (traffic/data/etc) that *flows* from sources to sinks.

Single Source Single Sink Flows

Simple setting:

- single source s and single sink t
- every other node v is an *internal* node
- flow originates at s and terminates at t



- Each edge e has a capacity $c(e) \geq 0$
- Some times it is convenient to assume that source $s \in V$ has no incoming edges and sink $t \in V$ has no outgoing edges

Assumptions: All capacities are integer, and every vertex has at least one edge incident to it.

Definition of Flow

Two ways to define flows:

- edge based
- path based

They are essentially equivalent but have different uses.

Edge based definition is more compact.

Edge Based Definition of Flow

Definition

A flow in a network $G = (V, E)$, is a function $f : E \to \mathbb{R}^{\geq 0}$ such that

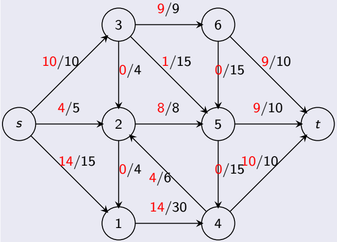


Figure: Flow with value

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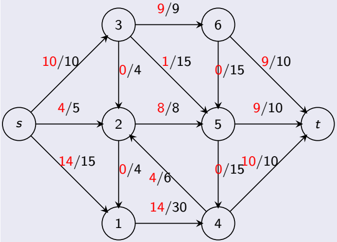

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- Capacity Constraint: For each edge e, $f(e) \leq c(e)$

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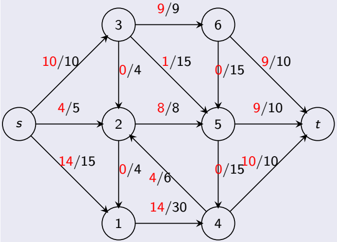

Figure: Flow with value

- Capacity Constraint: For each edge e, $f(e) \leq c(e)$
- Conservation Constraint: For each vertex $v \neq s, t$

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

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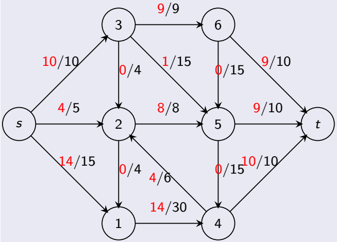

Figure: Flow with value 28

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$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

- Value of flow: (total flow out of source) $-$ (total flow in to source)

More Definitions and Notation

Notation

- The inflow into a vertex v is $f^{\rm in}(v) = \sum_{e \text{ into } v} f(e)$ and the outflow is $f^{\rm out}(v) = \sum_{e \text{ out of } v} f(e)$
- For a set of vertices A, $f^{\rm in}(A) = \sum_{e \text{ into } A} f(e)$. Outflow $f^{\rm out}(A)$ is defined analogously

Definition

For a network $G = (V, E)$ with source s, the value of flow f is defined as $v(f) = f^{\rm out}(s) - f^{\rm in}(s)$

A Path Based Definition of Flow

Intuition: flow goes from source s to sink t along a path.

$\mathcal{P}$: set of all paths from s to t. $|\mathcal{P}|$ can be *exponential* in n.

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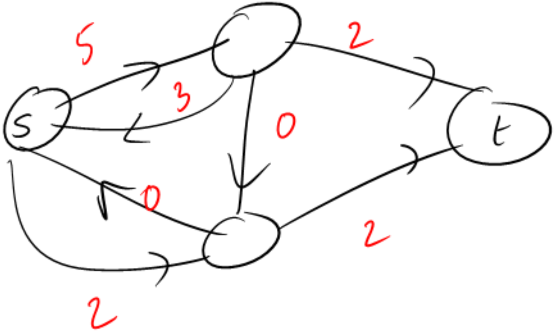

s
t
5
3
2
0
0
2
2

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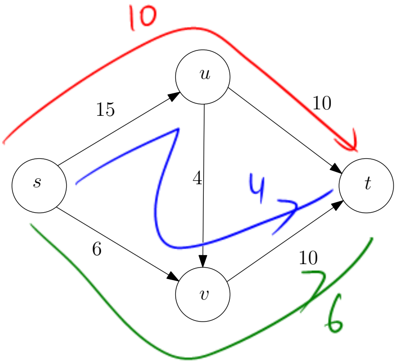
Value of flow: $\sum_{p \in \mathcal{P}} f(p)$

Example



$\mathcal{P} = \{p_1, p_2, p_3\}$
$p_1 : s \rightarrow u \rightarrow t$
$p_2 : s \rightarrow u \rightarrow v \rightarrow t$
$p_3 : s \rightarrow v \rightarrow t$

$f(p_1) = 10, f(p_2) = 4, f(p_3) = 6$

20

Path based flow implies Edge based flow

Lemma

Given a path based flow $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ there is an edge based flow $f' : E \rightarrow \mathbb{R}^{\geq 0}$ of the same value.

Path based flow implies Edge based flow

Lemma

Given a path based flow $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ *there is an edge based flow* $f' : E \rightarrow \mathbb{R}^{\geq 0}$ *of the same value.*

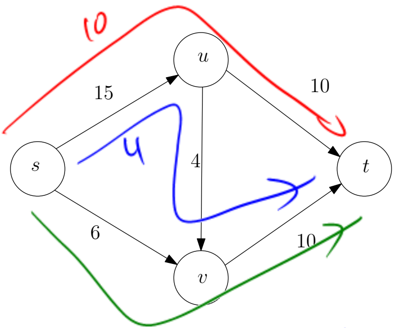
Proof.

For each edge e define $f'(e) = \sum_{p : e \in p} f(p)$.

Exercise: verify capacity and conservation constraints for f'.

Exercise: verify that value of f and f' are equal □

Example



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$p_1 : s \rightarrow u \rightarrow t$

$p_2 : s \rightarrow u \rightarrow v \rightarrow t$

$p_3 : s \rightarrow v \rightarrow t$

$f(p_1) = 10, f(p_2) = 4, f(p_3) = 6$

$f'((s, u)) = 14$

$f'((u, v)) = 4$

$f'((s, v)) = 6$

$f'((u, t)) = 10$

$f'((v, t)) = 10$

Flow Decomposition: Edge based flow to Path based Flow

Lemma

Given an edge based flow $f' : E \rightarrow \mathbb{R}^{\geq 0}$, *there is a path based flow* $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ *of same value. Moreover,* f *assigns non-negative flow to at most* m *paths where* $|E| = m$ *and* $|V| = n$. *Given* f', *the path based flow can be computed in* $O(mn)$ *time.*

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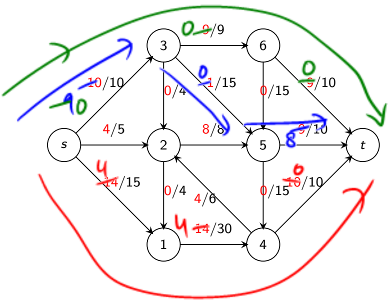
Proof Idea.

- remove all edges with $f'(e) = 0$
- find a path p from s to t
- assign $f(p)$ to be $\min_{e \in p} f'(e)$
- reduce $f'(e)$ for all $e \in p$ by $f(p)$
- repeat until no path from s to t
- in each iteration at least on edge has flow reduced to zero; hence at most m iterations. Can be implemented in $O(m(m+n))$ time. $O(mn)$ time requires care. □

Example



$P_1 = s \to 3 \to 5 \to t$

$f(P_1) = 1$

$P_2 = s \to 3 \to 6 \to t$

$f(P_2) = 9$

$P_3 = s \to 1 \to 4 \to t$

$f(P_3) = 10$

Edge vs Path based Definitions of Flow

Edge based flows:

- *compact* representation, only m values to be specified
- need to check flow conservation explicitly at each internal node

Path flows:

- in some applications, paths more natural
- not compact
- no need to check flow conservation constraints

Equivalence shows that we can go back and forth easily.

The Maximum-Flow Problem

Problem

Input: A network G with capacity c and source s and sink t

Goal: Find flow of *maximum* value

Problem

Input A network G with capacity c and source s and sink t

Goal Find flow of *maximum* value

Question: Given a flow network, what is an *upper bound* on the maximum flow between source and sink?

Cuts

Definition

Given a flow network an s-t cut is a set of edges $E' \subset E$ such that removing E' *disconnects* s from t: in other words there is no directed $s \rightarrow t$ path in $E - E'$.
The *capacity* of cut E' is $\sum_{e \in E'} c(e)$.

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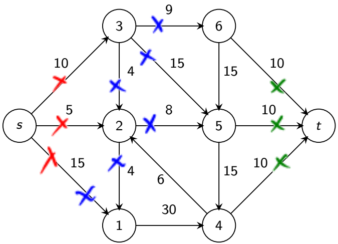

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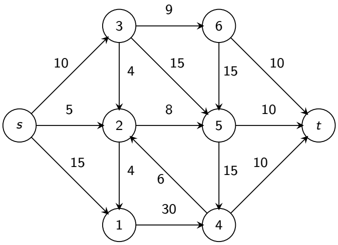
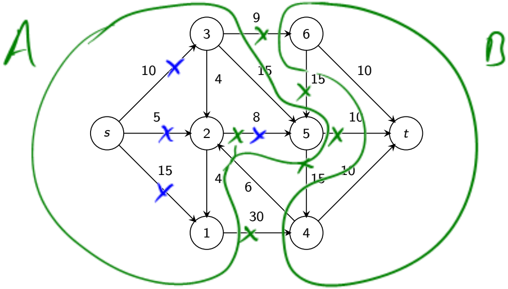


Caution: cut may leave $t \rightarrow s$ paths!

Minimal Cut

Definition

Given a flow network an s-t, E' is a *minimal* cut if for all $e \in E'$, $E' - \{e\}$ is not a cut.



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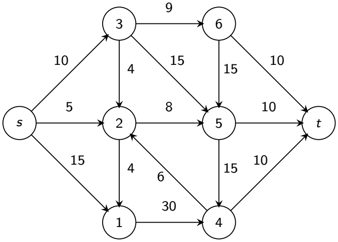


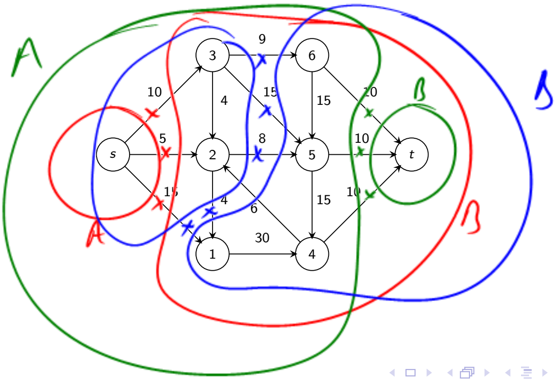
Observation: given a cut E', can check efficiently whether E' is a minimal cut or not. How?

Cuts as Vertex Partitions

Let $A \subset V$ such that

- $s \in A$, $t \notin A$
- $B = V - A$ and hence $t \in B$

Define cut $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$: edges leaving A



Cuts as Vertex Partitions

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Claim

(A, B) is an s-t cut.

Proof.

Let P be any $s \rightarrow t$ path in G. Since t is not in A, P has to leave A via some edge (u, v) in (A, B). □

Cuts as Vertex Partitions

Lemma

Suppose E' is an s-t cut. Then there is a cut (A, B) such that $(A, B) \subseteq E'$.

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Proof.

E' is an s-t cut implies no path from s to t in $(V, E - E')$.

- Let A be set of all nodes reachable by s in $(V, E - E')$.
- Since E' is a cut, $t \notin A$.
- $(A, B) \subseteq E'$. Why?

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Cuts as Vertex Partitions

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Corollary

Every minimal *s-t cut E' is a cut of the form (A, B).*

Minimum Cut

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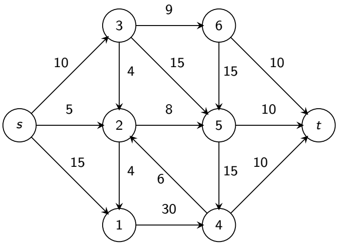
Given a flow network an *s*-*t minimum* cut is a cut E' of smallest capacity amongst all *s*-*t* cuts.



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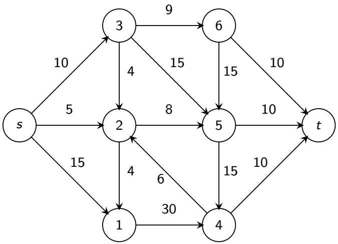

Observation: exponential number of *s*-*t* cuts and no "easy" algorithm to find a minimum cut.

The Minimum-Cut Problem

Problem

Input A flow network G

Goal Find the capacity of a *minimum* s-t cut

Flows and Cuts

Lemma

For any s-t cut E', maximum s-t flow $\leq$ capacity of E'.

Proof.

Formal proof easier with path based definition of flow.
Suppose $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ is a max-flow.

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Every path $p \in \mathcal{P}$ contains an edge $e \in E'$. Why?

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Every path $p \in \mathcal{P}$ contains an edge $e \in E'$. Why?
Assign each path $p \in \mathcal{P}$ to exactly one edge $e \in E'$.
Let $\mathcal{P}_e$ be paths assigned to $e \in E'$.

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Assign each path $p \in \mathcal{P}$ to exactly one edge $e \in E'$.
Let $\mathcal{P}_e$ be paths assigned to $e \in E'$. Then

$$v(f) = \sum_{p \in \mathcal{P}} f(p) = \sum_{e \in E'} \sum_{p \in \mathcal{P}_e} f(p) \leq \sum_{e \in E'} c(e)$$

□

Flows and Cuts

Lemma

For any s-t cut E', maximum *s-t flow* $\leq$ *capacity of* E'.

Corollary

Maximum s-t flow $\leq$ *minimum s-t cut.*

Max-Flow Min-Cut Theorem

Theorem

In any flow network the maximum s-t flow is equal *to the minimum s-t cut.*

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Can compute minimum-cut from maximum flow and vice-versa!

Max-Flow Min-Cut Theorem

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Can compute minimum-cut from maximum flow and vice-versa!
Proof coming shortly.

Many applications:

- optimization
- graph theory
- combinatorics

The Maximum-Flow Problem

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Input A network G with capacity c and source s and sink t

Goal Find flow of *maximum* value from s to t

The Maximum-Flow Problem

Problem

Input	A network G with capacity c and source s and sink t
Goal	Find flow of *maximum* value from s to t

Exercise: Given G, s, t as above, show that one can remove all edges into s and all edges out of t without affecting the flow value between s and t.