CS 473: Algorithms

Chandra Chekuri
chekuri@cs.illinois.edu
3228 Siebel Center

University of Illinois, Urbana-Champaign

Fall 2010

Part I

Hash Tables

Dictionary Data Structure

- A universe $\mathcal{U}$ of keys that have a total order: numbers, strings, etc.
- Data structure to store a subset $S \subseteq \mathcal{U}$
- **Operations:**
    - Search/lookup: given $x \in \mathcal{U}$ is $x \in S$?
    - Insert: given $x \notin S$ add x to S.
    - Delete: given $x \in S$ delete x from S
- *Static* strucutre: S given in advance or changes very infrequently, main operations are lookups.
- *Dynamic* structure: S changes rapidly so inserts and deletes as important as lookups.

Dictionary Data Structures

Common solutions:

- Static:
  - Store S as a *sorted* array
  - Lookup: binary search in $O(\log |S|)$ time (comparisons)
- Dyanamic:
  - Store S in a *balanced* binary search tree
  - Lookup, Insert, Delete in $O(\log |S|)$ time (comparisons)

Dictionary Data Structures

Question: "Should Tables be Sorted?"
(also title of famous paper by Turing award winner Andy Yao)

Dictionary Data Structures

Question: "Should Tables be Sorted?"
(also title of famous paper by Turing award winner Andy Yao)

Hashing is a widely used & powerful technique for dictionaries.

Motivation:

- Universe $\mathcal{U}$ may not be (naturally) totally ordered
- Keys correspond to large objects (images, graphs etc) for which comparisons are very expensive
- Want to improve "average" performance of lookups to $O(1)$ even at cost of extra space or errors with small probability: many applications for fast lookups in networking, security, etc.

Hashing and Hash Tables

Hash Table data structure:

- A (hash) table/array T of size m (the table size)
- A hash function $h : \mathcal{U} \to \{0, \ldots, m-1\}$
- Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in T

Hashing and Hash Tables

Hash Table data structure:

- A (hash) table/array T of size m (the table size)
- A hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m-1\}$
- Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in T

Given $S \subseteq \mathcal{U}$. How do we store S and how do we do lookups?

Hashing and Hash Tables

Hash Table data structure:

- A (hash) table/array T of size m (the table size)
- A hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m-1\}$
- Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in T

Given $S \subseteq \mathcal{U}$. How do we store S and how do we do lookups?

Ideal situation:

- Each element $x \in S$ hashes to a distinct slot in T. Store x in slot $h(x)$
- Lookup: given $y \in \mathcal{U}$ check if $T[h(y)] = y$. $O(1)$ time!

Hashing and Hash Tables

Hash Table data structure:

- A (hash) table/array T of size m (the table size)
- A hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m-1\}$
- Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in T

Given $S \subseteq \mathcal{U}$. How do we store S and how do we do lookups?

Ideal situation:

- Each element $x \in S$ hashes to a distinct slot in T. Store x in slot $h(x)$
- Lookup: given $y \in \mathcal{U}$ check if $T[h(y)] = y$. $O(1)$ time!

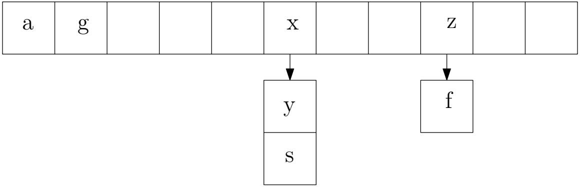
Collisions unavoidable. Several different techniques to handle them.

Handling Collisions: Chaining

Collision: $h(x) = h(y)$ for some $x \neq y$.

Chaining to handle collisions:

- For each slot i store all items hashed to slot i in a linked list. $T[i]$ poits to the linked list
- Lookup: to find if $y \in \mathcal{U}$ is in T, check the linked list at $T[h(y)]$. Time proportion to size of linked list.



Handling Collisions

Several other techniques:

- Open addressing
- . . .
- Cuckoo hashing

Understanding Hashing

Does hashing give $O(1)$ time per operation for dictionaries?

Understanding Hashing

Does hashing give $O(1)$ time per operation for dictionaries?

Questions:

- Complexity of evaluating h on a given element?
- Relative sizes of the universe $\mathcal{U}$ and the set to be stored S.
- Size of table relative to size of S.
- Worst-case vs average-case vs randomized (expected) time?
- How do we choose h?

Understanding Hashing

- Complexity of evaluating h on a given element? Should be small.
- Relative sizes of the universe $\mathcal{U}$ and the set to be stored S: typically $|\mathcal{U}| \gg |S|$.
- Size of table relative to size of S. The *load factor* of T is the ratio n/m where $n = |S|$. Typically n/m is a small constant greater than 1 (close to 2).

Understanding Hashing

- Complexity of evaluating h on a given element? Should be small.
- Relative sizes of the universe $\mathcal{U}$ and the set to be stored S: typically $|\mathcal{U}| \gg |S|$.
- Size of table relative to size of S. The *load factor* of T is the ratio n/m where $n = |S|$. Typically n/m is a small constant greater than 1 (close to 2).

Main and interrelated questions:

- Worst-case vs average-case vs randomized (expected) time?
- How do we choose h?

Single hash function

- Assume $N = |\mathcal{U}| \gg m$ where m is size of table T. In particular assume $N \geq m^2$ (very conservative).
- Fix hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m-1\}$.
- N items hashed to m slots. By pigeon hole principle there is some $i \in \{0, \ldots, m-1\}$ such that m elements of $\mathcal{U}$ get hashed to i!
- Implies that there is a set $S \subseteq \mathcal{U}$ where $|S| = m$ such that all of S hashes to same slot!

Single hash function

- Assume $N = |\mathcal{U}| \gg m$ where m is size of table T. In particular assume $N \geq m^2$ (very conservative).
- Fix hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m-1\}$.
- N items hashed to m slots. By pigeon hole principle there is some $i \in \{0, \ldots, m-1\}$ such that m elements of $\mathcal{U}$ get hashed to i!
- Implies that there is a set $S \subseteq \mathcal{U}$ where $|S| = m$ such that all of S hashes to same slot!

Lesson: For every hash function there is a very bad set!

Picking a hash function

- Hash function are often chosen in an ad hoc fashion. Implicit assumption is that input behaves well.
- Theory and sound practice suggests that a hash function should be chosen properly with guarantees on its behavior.

Picking a hash function

- Hash function are often chosen in an ad hoc fashion. Implicit assumption is that input behaves well.
- Theory and sound practice suggests that a hash function should be chosen properly with guarantees on its behavior.

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

- $\mathcal{H}$ is a *family* of hash functions: each function $h \in \mathcal{H}$ should be efficient to evaluate (that is, to compute $h(x)$)
- h is chosen *randomly* from $\mathcal{H}$ (typically uniformly at random). Implicitly assumes that $\mathcal{H}$ allows an efficient sampling.
- Randomized guarantee: should have the property that for any *fixed* set $S \subseteq \mathcal{U}$ of size m the expected number of collisions for a function chosen from $\mathcal{H}$ should be "small". Here the expectation is over the randomness in choice of h.

Picking a hash function

Question: Why not let $\mathcal{H}$ be the set of *all* functions from $\mathcal{U}$ to $\{0, 1, \ldots, m-1\}$?

Picking a hash function

Question: Why not let $\mathcal{H}$ be the set of *all* functions from $\mathcal{U}$ to $\{0, 1, \ldots, m-1\}$?

- Too many functions! A random function has high complexity!

Picking a hash function

Question: Why not let $\mathcal{H}$ be the set of *all* functions from $\mathcal{U}$ to $\{0, 1, \ldots, m-1\}$?

- Too many functions! A random function has high complexity!

Question: Are there good and compact families $\mathcal{H}$?

Picking a hash function

Question: Why not let $\mathcal{H}$ be the set of *all* functions from $\mathcal{U}$ to $\{0, 1, \ldots, m-1\}$?

- Too many functions! A random function has high complexity!

Question: Are there good and compact families $\mathcal{H}$?

- Yes, but one has to define what it means for $\mathcal{H}$ to be good and compact.

Uniform hashing

Question: What are good properties of $\mathcal{H}$ in distributing data?

Uniform hashing

Question: What are good properties of $\mathcal{H}$ in distributing data?

- Consider any element $x \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then x should go into a random slot in T. In other words $\Pr[h(x) = i] = 1/m$ for every $0 \leq i < m$.

Uniform hashing

Question: What are good properties of $\mathcal{H}$ in distributing data?

- Consider any element $x \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then x should go into a random slot in T. In other words $\Pr[h(x) = i] = 1/m$ for every $0 \leq i < m$.
- Consider any two distinct elements $x, y \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then the probability of a collision between x and y should be at most $1/m$. In other words $\Pr[h(x) = h(y)] = 1/m$ (cannot be smaller).

Uniform hashing

Question: What are good properties of $\mathcal{H}$ in distributing data?

- Consider any element $x \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then x should go into a random slot in T. In other words $\Pr[h(x) = i] = 1/m$ for every $0 \leq i < m$.
- Consider any two distinct elements $x, y \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then the probability of a collision between x and y should be at most $1/m$. In other words $\Pr[h(x) = h(y)] = 1/m$ (cannot be smaller).
- Second property is stronger than the first and the crucial issue.

Definition

A family hash function $\mathcal{H}$ is (2)-universal if for all distinct $x, y \in \mathcal{U}$, $\Pr[h(x) = h(y)] = 1/m$ where m is the table size.

Uniform hashing

Question: What are good properties of $\mathcal{H}$ in distributing data?

- Consider any element $x \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then x should go into a random slot in T. In other words $\Pr[h(x) = i] = 1/m$ for every $0 \leq i < m$.
- Consider any two distinct elements $x, y \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then the probability of a collision between x and y should be at most $1/m$. In other words $\Pr[h(x) = h(y)] = 1/m$ (cannot be smaller).
- Second property is stronger than the first and the crucial issue.

Definition

A family hash function $\mathcal{H}$ is (2)-universal if for all distinct $x, y \in \mathcal{U}$, $\Pr[h(x) = h(y)] = 1/m$ where m is the table size.

Note: The set of all hash function satisfies stronger properties!

Analyzing Uniform Hashing

- T is hash table of size m.
- $S \subseteq \mathcal{U}$ is a *fixed* set of size m.
- h is chosen randomly from uniform hash family $\mathcal{H}$.
- x is a *fixed* element of $\mathcal{U}$. Assume for simplicity that $x \notin S$.

Question: What is the *expected* time to look up x in T using h assuming chaining used to resolve collisions?

Analyzing Uniform Hashing

Question: What is the *expected* time to look up x in T using h assuming chaining used to resolve collisions?

- The time to look up x is the size of the list at $T[h(x)]$: same as the number of elements in S that collide with x under h.
- Let $\ell(x)$ be this number. We want $\mathrm{E}[\ell(x)]$
- For $y \in S$ let A_y be the even that x, y collide and D_y be the corresponding indicator variable.

$$\ell(x) = \sum_{y \in S} D_y$$

$$\begin{aligned} \Rightarrow \mathrm{E}[\ell(x)] &= \sum_{y \in S} \mathrm{E}[D_y] \quad \text{linearity of expectation} \\ &= \sum_{y \in S} Pr[h(x) = h(y)] = \sum_{y \in S} \frac{1}{m} \quad \text{since } \mathcal{H} \text{ is a uniform} \\ &= |S|/m \leq 1 \end{aligned}$$

Analyzing Uniform Hashing

Question: What is the *expected* time to look up x in T using h assuming chaining used to resolve collisions?

Answer: $O(1)$!

Analyzing Uniform Hashing

Question: What is the *expected* time to look up x in T using h assuming chaining used to resolve collisions?

Answer: $O(1)$!

Comments:

- $O(1)$ expected time also holds for insertion.
- Analysis assumes static set S but holds as long as S is a set formed with at most $O(m)$ insertions and deletions.
- *Worst-case* look up time can be large! How large? $\Omega(\log n / \log\log n)$.

Rehashing, amortization and making structure dynamic

Previous analysis assumed fixed S of size $\simeq m$.

Question: What happens as items are inserted and deleted?

- If $|S|$ grows to more than cm for some constant c then hast table performance clearly degrades
- If $|S|$ stays around $\simeq m$ but incurs many insertions and deletions then the initial random hash function is no longer random enough!

Rehashing, amortization and making structure dynamic

Previous analysis assumed fixed S of size $\simeq m$.

Question: What happens as items are inserted and deleted?

- If $|S|$ grows to more than cm for some constant c then hast table performance clearly degrades
- If $|S|$ stays around $\simeq m$ but incurs many insertions and deletions then the initial random hash function is no longer random enough!

Solution: Rebuild hash table periodically!

- Choose a new table size based on current number of elements in table.
- Choose a new random hash function and rehash the elements.
- Discard old table and hash function.

Question: When to rebuild? How expensive?

Rebuilding the hash table

- Start with table size m where m is some estimate of $|S|$ (can be some large constant).
- If $|S|$ grows to more than twice current table size, build new hash table (choose a new random hash function) with double the current number of elements. Can also use similar trick if table size falls below quarter the size.
- If $|S|$ stays roughly the same but more than $c|S|$ operations on table for some chosen constant c (say 10), rebuild.

Amortize cost of rebuilding to previously performed operations. Rebuilding ensures $O(1)$ expected analysis holds even when S changes. Hence $O(1)$ expected look up/insert/delete time *dynamic* data dictionary data structure!

Constructing Universal Hash Families

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

- Choose prime number $p \geq N$. $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ is a field.
- For $a, b \in \mathbb{Z}_p$, $a \neq 0$, define the hash function $h_{a,b}$ as $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
- Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0$. Note that $|\mathcal{H}| = p(p-1)$.

Constructing Universal Hash Families

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

- Choose prime number $p \geq N$. $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ is a field.
- For $a, b \in \mathbb{Z}_p$, $a \neq 0$, define the hash function $h_{a,b}$ as $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
- Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0$. Note that $|\mathcal{H}| = p(p-1)$.

Theorem

$\mathcal{H}$ is a (2)-universal hash family.

Constructing Universal Hash Families

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

- Choose prime number $p \geq N$. $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ is a field.
- For $a, b \in \mathbb{Z}_p$, $a \neq 0$, define the hash function $h_{a,b}$ as $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
- Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0$. Note that $|\mathcal{H}| = p(p-1)$.

Theorem

$\mathcal{H}$ is a (2)-universal hash family.

Comments:

- Hash family is of small size, easy to sample from.
- Easy to store a hash function (a, b have to be stored) and evaluate it.

Constructing Universal Hash Families

- Choose prime number $p \geq N$. $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ is a field.
- For $a, b \in \mathbb{Z}_p$, $a \neq 0$, define the hash function $h_{a,b}$ as $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
- Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0$. Note that $|\mathcal{H}| = p(p-1)$.

Constructing Universal Hash Families

- Choose prime number $p \geq N$. $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ is a field.
- For $a, b \in \mathbb{Z}_p$, $a \neq 0$, define the hash function $h_{a,b}$ as $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
- Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0$. Note that $|\mathcal{H}| = p(p-1)$.

Theorem

$\mathcal{H}$ is a (2)-universal hash family.

Proof.

Fix $x, y \in \mathcal{U}$. What is the probability they will collide if h is picked randomly from $\mathcal{H}$?

Constructing Universal Hash Families

- Choose prime number $p \geq N$. $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ is a field.
- For $a, b \in \mathbb{Z}_p$, $a \neq 0$, define the hash function $h_{a,b}$ as $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
- Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0$. Note that $|\mathcal{H}| = p(p-1)$.

Theorem

$\mathcal{H}$ is a (2)-universal hash family.

Proof.

Fix $x, y \in \mathcal{U}$. What is the probability they will collide if h is picked randomly from $\mathcal{H}$?

- Let a, b be bad for x, y if $h_{a,b}(x) = h_{a,b}(y)$
- **Claim:** Number of bad pairs is at most $p(p-1)/m$.
- Total number of hash functions is $p(p-1)$ and hence probability of a collision is $\leq 1/m$. ☐

Some Lemmas

Lemma

If $x \neq y$ then for any $a, b \in \mathbb{Z}_p$ such that $a \neq 0$, $ax + b \mod p \neq ay + b \mod p$.

Proof.

If $ax + b \mod p = ay + b \mod p$ then $a(x - y) \mod p = 0$ and $a \neq 0$ and $(x - y) \neq 0$. However, a and $(x - y)$ cannot divide p since p is prime and $a, (x - y) < p$. ☐

Some Lemmas

Lemma

If $x \neq y$ then for each (r, s) such that $r \neq s$ and $0 \leq r, s \leq p - 1$ there is exactly one a, b such that $ax + b \mod p = r$ and $ay + b \mod p = s$.

Proof.

Solve the two equations:

$$ax + b = r \mod p \text{ and } ay + b = s \mod p$$

We get $a = \frac{r-s}{x-y} \mod p$ and $b = r - ax \mod p$. ☐

Proof of Claim

Proof.

Let $a, b \in \mathbb{Z}_p$ such that $a \neq 0$ and $h_{a,b}(x) = h_{a,b}(y)$.

- Let $ax + b \mod p = r$ and $ay + b \mod = s \mod p$.
- Collision if and only if $r = s \mod m$.
- Number of pairs (r, s) such that $r \neq s$ and $0 \leq r, s \leq p - 1$ and $r = s \mod m$ is $p(p-1)/m$.
- From previous lemma for each bad pair (a, b) there is a unique pair (r, s) such that $r = s \mod m$. Hence total number of bad pairs is $p(p-1)/m$.

☐

Perfect Hashing

Question: Can we make look up time $O(1)$ in worst case?

Yes for static dictionaries but then space usage is $O(m)$ only in expectation.

Take away points

- Hashing is a powerful and important technique for dictionaries. Many practical applications.
- Randomization fundamental to understanding hashing.
- Good and efficient hashing possible in theory and practice with proper definitions (universal, perfect, etc).
- Related ideas of creating a compact fingerprint/sketch for objects is very powerful in theory and practice.
- Many applications in practice.