CS 473: Algorithms

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Fall 2010
Part I

Problems and Terminology
Problem Types

- **Decision Problem:** Is the input a YES or NO input?
  Example: Given graph $G$, nodes $s$, $t$, is there a path from $s$ to $t$ in $G$?

- **Search Problem:** Find a *solution* if input is a YES input.
  Example: Given graph $G$, nodes $s$, $t$, find an $s$-$t$ path.

- **Optimization Problem:** Find a *best* solution among all solutions for the input.
  Example: Given graph $G$, nodes $s$, $t$, find a shortest $s$-$t$ path.
A problem $\Pi$ consists of an *infinite* collection of inputs \( \{ l_1, l_2, \ldots, \} \). Each input is referred to as an *instance*.

The *size* of an instance $l$ is the number of bits in its representation.

For an instance $l$, $\text{sol}(l)$ is a set of *feasible solutions* to $l$. *Typical implicit assumption:* given instance $l$ and $y \in \Sigma^*$, there is an way to check if $y \in \text{sol}(l)$. In other words, problem is in NP.

For optimization problems each solution $s \in \text{sol}(l)$ has an associated *value*. *Typical implicit assumption:* given $s$, can compute value efficiently.
Problem Types

- **Decision Problem**: Given $I$ output whether $sol(I) = \emptyset$ or not.
- **Search Problem**: Given $I$, find a solution $s \in sol(I)$ if $sol(I) \neq \emptyset$.
- **Optimization Problem**: Given $I$,
  - Minimization problem. Find a solution $s \in sol(I)$ of minimum value
  - Maximization problem. Find a solution $s \in sol(I)$ of maximum value
  - Notation: $opt(I)$: interchangeably (when there is no confusion) used to denote the value of an optimum solution or some fixed optimum solution.
Part II

Greedy Algorithms: Tools and Techniques
What is a Greedy Algorithm?

No real consensus on a universal definition.

Greedy algorithms:
- make decision incrementally in small steps without backtracking
- decision at each step is based on improving local or current state in a myopic fashion without paying attention to the global situation
- decisions often based on some fixed and simple priority rules
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Pros and Cons of Greedy Algorithms

Pros:

- Usually (too) easy to design greedy algorithms
- Easy to implement and often run fast since they are simple
- Several important cases where they are effective/optimal
- Lead to a first-cut heuristic when problem not well understood

Cons:

- Very often greedy algorithms don’t work. Easy to lull oneself into believing they work
- Many greedy algorithms possible for a problem and no structured way to find effective ones
- CS 473: Every greedy algorithm needs a proof of correctness
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Greedy Algorithm Types

Crude classification:

- **Non-adaptive**: fix some ordering of decisions apriori and stick with the order
- **Adaptive**: make decisions adaptively but greedily/locally at each step
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Plan:
- See several examples
- Pick up some proof techniques
Interval Scheduling

**Input**  A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms)

**Goal**  Schedule as many jobs as possible
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- Two jobs with overlapping intervals cannot both be scheduled!

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![Diagram of intervals]
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- Two jobs with overlapping intervals cannot both be scheduled!
Greedy Template

Initially R is the set of all requests
A is empty (* A will store all the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R
    add i to A
    remove from R all requests that overlap with i
return the set A
Greedy Template

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Main task: Decide the order in which to process requests in R
Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.
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Figure: Counter example for earliest start time
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Smallest Processing Time

Process jobs in the order of processing time, starting with jobs that require the shortest processing.
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Fewest Conflicts

Process jobs in that have the fewest “conflicts” first.
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Optimal Greedy Algorithm

Initially R is the set of all requests
A is empty (* A will store all the jobs that will be scheduled *)
while R is not empty
    choose $i \in R$ such that finishing time of $i$ is least
    add $i$ to A
    remove from R all requests that overlap with $i$
return the set A

Theorem

The greedy algorithm that picks jobs in the order of their finishing times is optimal.
Proving Optimality

- **Correctness**: Clearly the algorithm returns a set of jobs that does not have any conflicts.
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For a set of requests $R$, let $O$ be an optimal set and let $A$ be the set returned by the greedy algorithm. Then $O = A$? Not likely!

---

![Diagram showing intervals for Proving Optimality](image_url)
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Instead we will show that $|O| = |A|$.
Proof of Optimality: Key Lemma

Lemma

Let $i_1$ be first interval picked by Greedy. There exists an optimum solution that contains $i_1$.

Proof.

Let $O$ be an arbitrary optimum solution. If $i_1 \in O$ we are done.
Proof of Optimality: Key Lemma

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Let $O$ be an arbitrary optimum solution. If $i_1 \in O$ we are done.

**Claim:** If $i_1 \not\in O$ then there is exactly one interval $j_1 \in O$ that conflicts with $i_1$. (proof later)

- Form a new set $O'$ by removing $j_1$ from $O$ and adding $i_1$, that is $O' = (O - \{j_1\}) \cup \{i_1\}$.
- From claim, $O'$ is a feasible solution (no conflicts).
- Since $|O'| = |O|$, $O'$ is also an optimum solution and it contains $i_1$. 

Proof complete.
Proof of Claim

Claim

If \( i_1 \notin O \) then there is exactly one interval \( j_1 \in O \) that conflicts with \( i_1 \).

Proof.

- Suppose \( j_1, j_2 \in O \) such that \( j_1 \neq j_2 \) and both \( j_1 \) and \( j_2 \) conflict with \( i_1 \).
- Since \( i_1 \) has earliest finish time, \( j_1 \) and \( i_1 \) overlap at \( f(i_1) \).
- For same reason \( j_2 \) also overlaps with \( i_1 \) at \( f(i_1) \).
- Implies that \( j_1, j_2 \) overlap at \( f(i_1) \) contradicting the feasibility of \( O \).

See figure in next slide.
Figure for proof of Claim

Figure: Since $i_1$ has the earliest finish time, any interval that conflicts with it does so at $f(i_1)$. This implies $j_1$ and $j_2$ conflict.
Proof of Optimality of Earliest Finish Time First

Proof by Induction on number of intervals.

Base Case: \( n = 1 \). Trivial since Greedy picks one interval.

Induction Step: Assume theorem holds for \( i < n \).

Let \( I \) be an instance with \( n \) intervals.

\( I' \): \( I \) with \( i_1 \) and all intervals that overlap with \( i_1 \) removed.

\( G(I), G(I') \): Solution produced by Greedy on \( I \) and \( I' \).

From Lemma, there is an optimum solution \( O \) to \( I \) and \( i_1 \in O \).

Let \( O' = O - \{i_1\} \). \( O' \) is a solution to \( I' \).

\[
\begin{align*}
|G(I)| &= 1 + |G(I')| \quad \text{(from Greedy description)} \\
&\geq 1 + |O'| \quad \text{(By induction, \( G(I') \) is optimum for \( I' \))} \\
&= |O|
\end{align*}
\]
Implementation and Running Time

Initially $R$ is the set of all requests
A is empty (* A will store all the jobs that will be scheduled *)
while $R$ is not empty
  choose $i \in R$ such that finishing time of $i$ is least
  add $i$ to $A$
  remove from $R$ all requests that overlap with $i$
return the set $A$

Pre-sort all requests based on finishing time.
$O(n \log n)$ time

Now choosing least finishing time is $O(1)$
Keep track of the finishing time of the last request added to $A$. Then check if starting time of $i$ later than that
Thus, checking non-overlapping is $O(1)$

Total time $O(n \log n + n) = O(n \log n)$
Implementation and Running Time

Initially R is the set of all requests
A is empty (* A will store all the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R such that finishing time of i is least
    if i does not overlap with requests in A
        add i to A
return the set A
Initially \( R \) is the set of all requests
\( A \) is empty (* \( A \) will store all the jobs that will be scheduled *)
while \( R \) is not empty
    choose \( i \in R \) such that finishing time of \( i \) is least
    if \( i \) does not overlap with requests in \( A \)
        add \( i \) to \( A \)
return the set \( A \)

- Pre-sort all requests based on finishing time. \( O(n \log n) \) time
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Comments

- Interesting Exercise: smallest interval first picks at least half the optimum number of intervals.

- Instead of maximizing the total number of requests, associate *value/weight* with each job that is scheduled. Try to schedule jobs to maximize total value/weight. No greedy algorithm. Will be seen later in this course to illustrate dynamic programming.

- All requests need not be known at the beginning. Such *online* algorithms are a subject of research.
Scheduling all Requests

**Input**  A set of lectures, with start and end times

**Goal**  Find the minimum number of classrooms, needed to schedule all the lectures such two lectures do not occur at the same time in the same room.

![Diagram of lectures and classrooms](image)

**Figure:** A schedule requiring 4 classrooms
Scheduling all Requests

**Input**  A set of lectures, with start and end times

**Goal**  Find the minimum number of classrooms, needed to schedule all the lectures such two lectures do not occur at the same time in the same room.

**Figure:** A schedule requiring 4 classrooms

**Figure:** A schedule requiring 3 classrooms
Greedy Algorithm

Initially R is the set of all requests
d = 0 (* number of classrooms *)
while R is not empty
  choose i ∈ R
  if i can be scheduled in some class-room k ≤ d
    schedule lecture i in class-room k
  else
    allocate a new class-room d+1 and schedule lecture i in d+1
  d = d+1

What order should we process requests in?
Greedy Algorithm

Initially R is the set of all requests
d = 0 (* number of classrooms *)
while R is not empty
    choose i ∈ R such that start time of i is earliest
    if i can be scheduled in some class-room k ≤ d
        schedule lecture i in class-room k
    else
        allocate a new class-room d+1 and schedule lecture i in d+1
    d = d+1

What order should we process requests in? According to start times (breaking ties arbitrarily)
Depth of Lectures

**Definition**

For a set of lectures $R$, $k$ are said to be in conflict if there is some time $t$ such that there are $k$ lectures going on at time $t$. 
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- The depth of a set of lectures $R$ is the maximum number of lectures in conflict at any time.
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\[
\begin{array}{c}
& & & c & & d & & f & & j \\
& & b & & & g & & i & & \\
& & a & & e & & h & & \\
\end{array}
\]
Depth and Number of Class-rooms

Lemma

For any set $R$ of lectures, the number of class-rooms required is at least the depth of $R$. 
Depth and Number of Class-rooms

**Lemma**

For any set $R$ of lectures, the number of class-rooms required is at least the depth of $R$.

**Proof.**

All lectures that are in conflict must be scheduled in different rooms.
Lemma

Let \( d \) be the depth of the set of lectures \( R \). The number of class-rooms used by the greedy algorithm is \( d \).

Proof.

Suppose the greedy algorithm uses more than \( d \) rooms. Let \( j \) be the first lecture that is scheduled in room \( d + 1 \).
Lemma

Let $d$ be the depth of the set of lectures $R$. The number of class-rooms used by the greedy algorithm is $d$.

Proof.

- Suppose the greedy algorithm uses more than $d$ rooms. Let $j$ be the first lecture that is scheduled in room $d + 1$.
- Since we process lectures according to start times, there are $d$ lectures that start (at or) before $j$ and which are in conflict with $j$. 
Number of Class-rooms used by Greedy Algorithm

**Lemma**

Let \( d \) be the depth of the set of lectures \( R \). The number of class-rooms used by the greedy algorithm is \( d \).

**Proof.**

- Suppose the greedy algorithm uses more than \( d \) rooms. Let \( j \) be the first lecture that is scheduled in room \( d + 1 \).
- Since we process lectures according to start times, there are \( d \) lectures that start (at or) before \( j \) and which are in conflict with \( j \).
- Thus, at the start time of \( j \), there are at least \( d + 1 \) lectures in conflict, which contradicts the fact that the depth is \( d \).
Figure

\[ j \]

\[ s(j) \]

--- no such job scheduled before \( j \)
Correctness

Observation

The greedy algorithm does not schedule two overlapping lectures in the same room.

Theorem

The greedy algorithm is correct and uses the optimal number of class-rooms.
Implementation and Running Time

Initially R is the set of all requests
d = 0 (* number of classrooms *)
while R is not empty
   choose i ∈ R such that start time of i is earliest
   if i can be scheduled in some class-room k ≤ d
      schedule lecture i in class-room k
   else
      allocate a new class-room d+1 and schedule lecture i in d+1
      d = d+1
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        \( d = d+1 \)

- Pre-sort according to start times. Picking lecture with earliest start time can be done in \( O(1) \) time.
Initially R is the set of all requests

\[ d = 0 \] (* number of classrooms *)

while R is not empty

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if i can be scheduled in some class-room \( k \leq d \)

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    choose $i \in R$ such that start time of $i$ is earliest
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        schedule lecture $i$ in class-room $k$
    else
        allocate a new class-room $d+1$ and schedule lecture $i$ in $d+1$
        $d = d+1$

- Pre-sort according to start times. Picking lecture with earliest start time can be done in $O(1)$ time.
- Keep track of the finish time of last lecture in each room.
- Checking conflict takes $O(d)$ time.
- Total time $= O(n \log n + nd)$
Implementations and Running Time

Initially $R$ is the set of all requests

$d = 0 \ (* \ number \ of \ classrooms \ *)$

while $R$ is not empty

choose $i \in R$ such that start time of $i$ is earliest

if $i$ can be scheduled in some class-room $k \leq d$

schedule lecture $i$ in class-room $k$

else

allocate a new class-room $d+1$ and schedule lecture $i$ in $d+1$

$d = d+1$

Pre-sort according to start times. Picking lecture with earliest start time can be done in $O(1)$ time.

Keep track of the finish time of last lecture in each room.

With priority queues, checking conflict takes $O(\log d)$ time.

Total time $= O(n \log n + n \log d) = O(n \log n)$
Scheduling to Minimize Lateness

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job \( i \) starts at time \( s_i \) then it will finish at time \( f_i = s_i + t_i \), where \( t_i \) is its processing time.
- The lateness of a job is \( l_i = \max(0, f_i - d_i) \).
- Schedule all jobs such that \( L = \max l_i \) is minimized.
Scheduling to Minimize Lateness

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<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>$t_i$</td>
<td>3</td>
<td>2</td>
<td>1</td>
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<tr>
<td>$d_i$</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

$\begin{array}{ccccccc}
3 & 2 & 6 & 1 & 5 & 4 \\
\end{array}$

- $l_1 = 2$
- $l_5 = 0$
- $l_4 = 6$
A Simpler Feasibility Problem

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job $i$ starts at time $s_i$ then it will finish at time $f_i = s_i + t_i$, where $t_i$ is its processing time.
- Schedule all jobs such that each of them completes before its deadline (in other words $L = \max_i l_i = 0$).

**Definition**

A schedule is **feasible** if all jobs finish before their deadline.
Initially R is the set of all requests
curr-time = 0
while R is not empty
    choose i ∈ R
    curr-time = curr-time + ti
    if (curr-time > di) then
        return ‘‘no feasible schedule’’
    end if
end while
return ‘‘found feasible schedule’’
Greedy Template

Initially \( R \) is the set of all requests
\[ \text{curr-time} = 0 \]
while \( R \) is not empty
\[
\begin{align*}
\text{choose } i & \in R \\
\text{curr-time} & = \text{curr-time} + t_i \\
\text{if } (\text{curr-time} > d_i) & \text{ then} \\
\quad & \text{return ‘‘no feasible schedule’’}
\end{align*}
\]
end while
return ‘‘found feasible schedule’’

**Main task:** Decide the order in which to process jobs in \( R \)
Three Algorithms

- Shortest job first — sort according to $t_i$.
- Shortest slack first — sort according to $d_i - t_i$.
- Earliest deadline first — sort according to $d_i$. 
Interval Scheduling

Interval Partitioning

Scheduling to Minimize Lateness

The Problem

The Algorithm

Three Algorithms

- Shortest job first — sort according to $t_i$.
- Shortest slack first — sort according to $d_i - t_i$.
- Earliest deadline first — sort according to $d_i$.

Counter examples for first two: exercise
Earliest Deadline First

Theorem

Greedy with EDF rule for picking requests correctly decides if there is a feasible schedule.
Theorem

*Greedy with EDF rule for picking requests correctly decides if there is a feasible schedule.*

Proof via an exchange argument.
Earliest Deadline First

**Theorem**

*Greedy with EDF rule for picking requests correctly decides if there is a feasible schedule.*

Proof via an exchange argument.

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Earliest Deadline First

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**Lemma**

*If there is a feasible schedule then there is one with no idle time before all jobs are finished.*
Inversions

**Definition**

A schedule $S$ is said to have an **inversion** if there are jobs $i$ and $j$ such that $S$ schedules $i$ before $j$, but $d_i > d_j$. 
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**Claim**

*If a schedule $S$ has an inversion then there is an inversion between two adjacently scheduled jobs.*

Proof: exercise.
Main Lemma

Lemma

If there is a feasible schedule, then there is one with no inversions.

Proof Sketch.

Let $S$ be a schedule with minimum number of inversions.

- If $S$ has 0 inversions, done.
- Suppose $S$ has one or more inversions. By claim there are two adjacent jobs $i$ and $j$ that define an inversion.
- Swap positions of $i$ and $j$.
- New schedule is still feasible. (Why?)
- New schedule has one fewer inversion — contradiction!
Goal: schedule to minimize $L = \max_i l_i$. 

How can we use algorithm for simpler feasibility problem?

Given a lateness bound $L$, can we check if there is a schedule such that $\max_i l_i \leq L$?

Yes! Set $d'_i = d_i + L$ for each job $i$. Use feasibility algorithm with new deadlines.

How can we find minimum $L$?

Binary search!
Goal: schedule to minimize $L = \max_i l_i$.

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How can we find minimum $L$? Binary search!
Binary search for finding minimum lateness

\[ L = L_{\text{min}} = 0 \]
\[ L_{\text{max}} = \sum_i t_i \quad // \text{why is this sufficient?} \]

While \( L_{\text{min}} < L_{\text{max}} \) do
  \[ L = \lfloor (L_{\text{max}} + L_{\text{min}})/2 \rfloor \]
  check if there is a feasible schedule with lateness \( L \)
  if ‘yes’ then \( L_{\text{max}} = L \)
  else \( L_{\text{min}} = L + 1 \)
endwhile

return \( L \)
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Running time: \( O(n \log n \cdot \log T) \) where \( T = \sum_i t_i \)
- \( O(n \log n) \) for feasibility test (sort by deadlines)
- \( O(\log T) \) calls to feasibility test in binary search
Do we need binary search?

What happens in each call?
EDF algorithm with deadlines $d'_i = d_i + L$. 
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EDF algorithm with deadlines \( d'_i = d_i + L \).

Greedy with EDF schedules the jobs in the same order for all \( L \)!!!

Maybe there is a direct greedy algorithm for minimizing maximum lateness?
Initially R is the set of all requests
curr-time = 0
curr-late = 0
while R is not empty
    choose i ∈ R with earliest deadline
    curr-time = curr-time + $t_i$
    late = curr-time - $d_i$
    curr-late = max (late, curr-late)
return curr-late
Greedy Algorithm for Minimizing Lateness

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Exercise: argue directly that above algorithm is correct (see book).
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Can be easily implemented in $O(n \log n)$ time after sorting jobs.
Greedy Analysis: Overview

- **Greedy’s first step leads to an optimum solution.** Show that there is an optimum solution leading from the first step of Greedy and then use induction. Example, Interval Scheduling.

- **Greedy algorithm stays ahead.** Show that after each step the solution of the greedy algorithm is at least as good as the solution of any other algorithm. Example, Interval scheduling.

- **Structural property of solution.** Observe some structural bound of every solution to the problem, and show that greedy algorithm achieves this bound. Example, Interval Partitioning.

- **Exchange argument.** Gradually transform any optimal solution to the one produced by the greedy algorithm, without hurting its optimality. Example, Minimizing lateness.
Takeaway Points

- Greedy algorithms come naturally but often are incorrect. A proof of correctness is an absolute necessity.

- *Exchange* arguments are often the key proof ingredient. Focus on why the first step of the algorithm is correct: need to show that there is an optimum/correct solution with the first step of the algorithm.

- Thinking about correctness is also a good way to figure out which of the many greedy strategies is likely to work.