# CS 473: Algorithms

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# Part I

## Problems and Terminology

# Problem Types

- Decision Problem: Is the input a YES or NO input?
  Example: Given graph $G$, nodes $s, t$, is there a path from $s$ to $t$ in $G$?
- Search Problem: Find a *solution* if input is a YES input.
  Example: Given graph $G$, nodes $s, t$, find an $s$-$t$ path.
- Optimization Problem: Find a *best* solution among all solutions for the input.
  Example: Given graph $G$, nodes $s, t$, find a shortest $s$-$t$ path.

# Terminology

- A problem $\Pi$ consists of an *infinite* collection of inputs $\{I_1, I_2, \ldots, \}$. Each input is referred to as an instance.
- The size of an instance $I$ is the number of bits in its representation.
- For an instance $I$, $sol(I)$ is a set of feasible solutions to $I$. *Typical implicit assumption:* given instance $I$ and $y \in \Sigma^*$, there is an way to check if $y \in sol(I)$. In other words, problem is in NP.
- For optimization problems each solution $s \in sol(I)$ has an associated value. *Typical implicit assumption:* given $s$, can compute value efficiently.

# Problem Types

- Decision Problem: Given $I$ output whether $sol(I) = \emptyset$ or not.
- Search Problem: Given $I$, find a solution $s \in sol(I)$ if $sol(I) \neq \emptyset$.
- Optimization Problem: Given $I$,
  - Minimization problem. Find a solution $s \in sol(I)$ of minimum value
  - Maximization problem. Find a solution $s \in sol(I)$ of maximum value
  - Notation: $opt(I)$: interchangeably (when there is no confusion) used to denote the value of an optimum solution or some fixed optimum solution.

# Part II

## Greedy Algorithms: Tools and Techniques

# What is a Greedy Algorithm?

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No real consensus on a universal definition.

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Greedy algorithms:

- make decision incrementally in small steps *without backtracking*
- decision at each step is based on improving *local or current* state in a myopic fashion without paying attention to the *global* situation
- decisions often based on some fixed and simple *priority* rules

# Pros and Cons of Greedy Algorithms

Pros:

- Usually (too) easy to design greedy algorithms
- Easy to implement and often run fast since they are simple
- Several important cases where they are effective/optimal
- Lead to a first-cut heuristic when problem not well understood

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- Many greedy algorithms possible for a problem and no structured way to find effective ones

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CS 473: Every greedy algorithm needs a proof of correctness

# Greedy Algorithm Types

Crude classification:

- Non-adaptive: fix some ordering of decisions apriori and stick with the order
- Adaptive: make decisions adaptively but greedily/locally at each step

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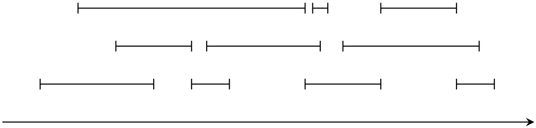
Plan:

- See several examples
- Pick up some proof techniques

# Interval Scheduling

**Input** A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms)

**Goal** Schedule as many jobs as possible



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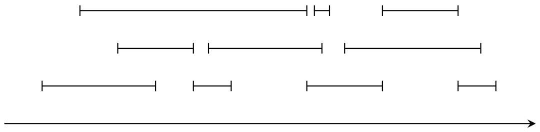

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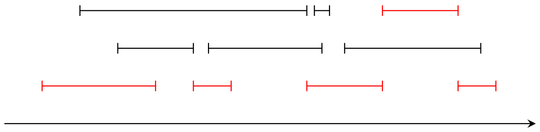

# Greedy Template

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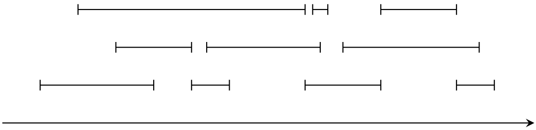
Main task: Decide the order in which to process requests in $R$

ES SP FC EF

# Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.



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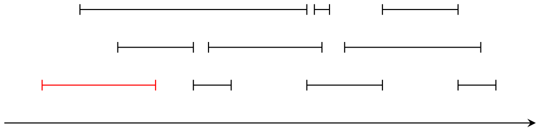

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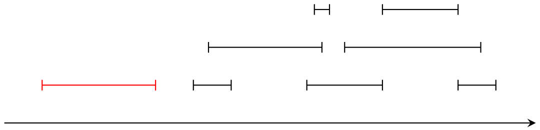

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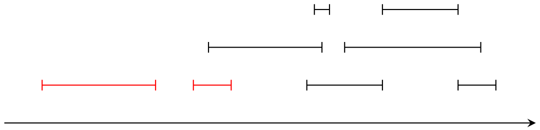

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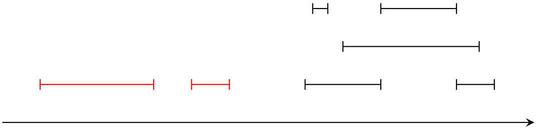

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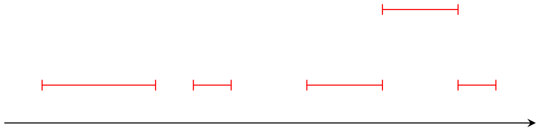

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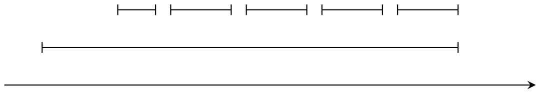

Figure: Counter example for earliest start time

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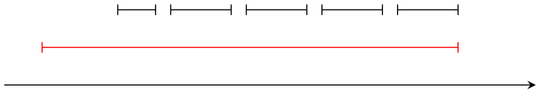

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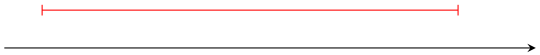

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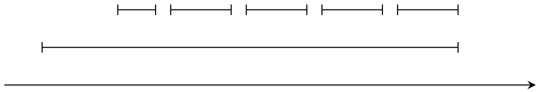
Process jobs in the order of processing time, starting with jobs that require the shortest processing.



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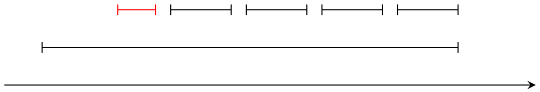

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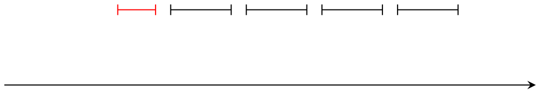

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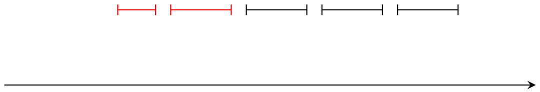

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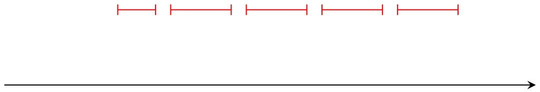

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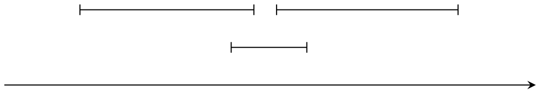

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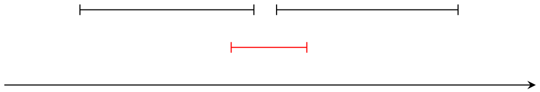

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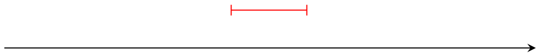

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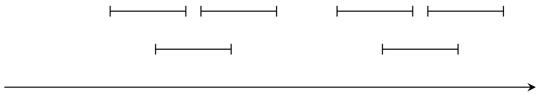
Process jobs in that have the fewest “conflicts” first.



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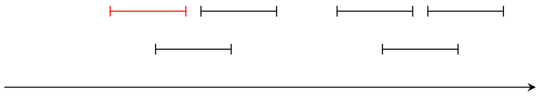

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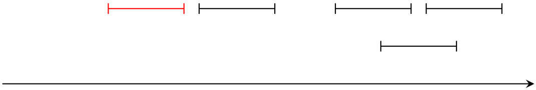

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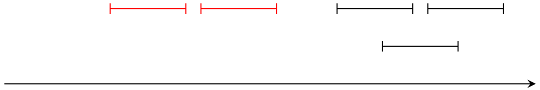

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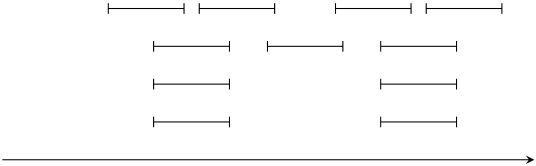

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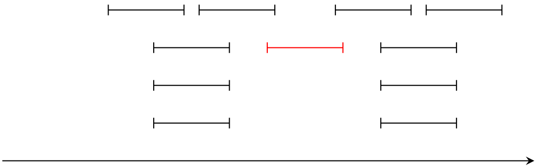

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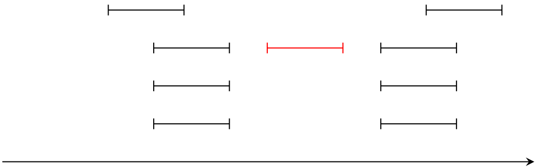

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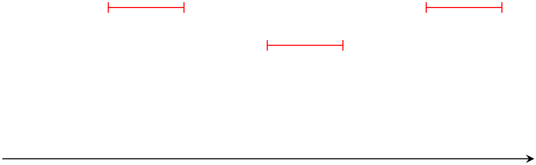

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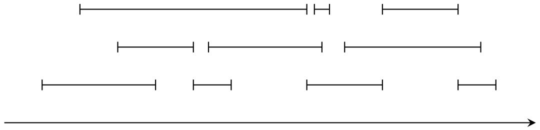

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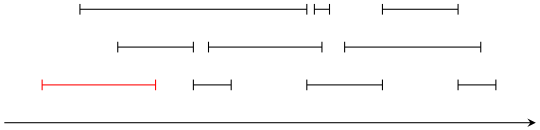

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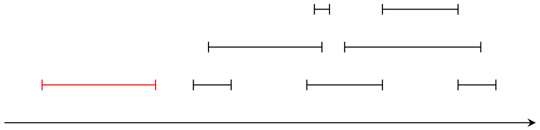

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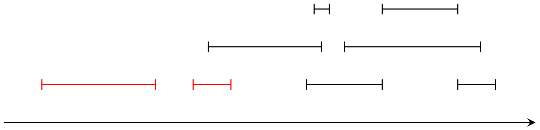

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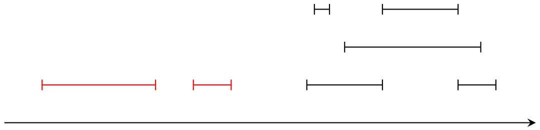

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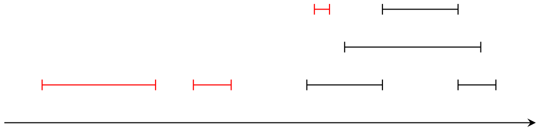

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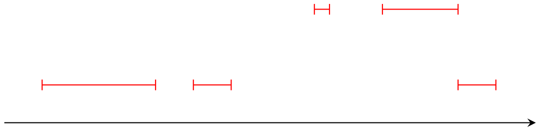

# Optimal Greedy Algorithm

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Initially R is the set of all requests
A is empty (* A will store all the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R such that finishing time of i is least
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**Theorem**

*The greedy algorithm that picks jobs in the order of their finishing times is optimal.*

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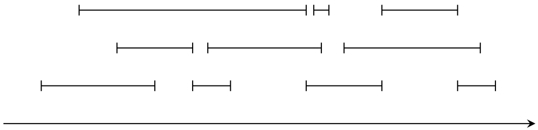

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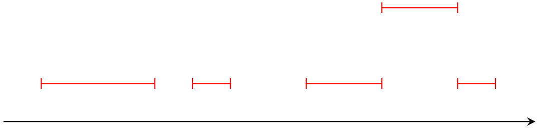

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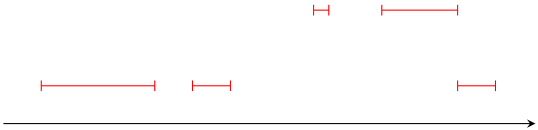

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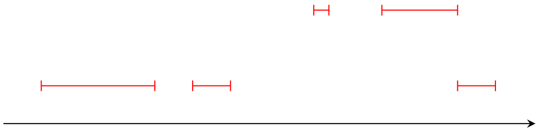

Instead we will show that $|O| = |A|$

# Proof of Optimality: Key Lemma

**Lemma**

*Let $i_1$ be first interval picked by Greedy. There exists an optimum solution that contains $i_1$.*

**Proof.**

Let $O$ be an *arbitrary* optimum solution. If $i_1 \in O$ we are done.

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**Claim:** If $i_1 \notin O$ then there is exactly one interval $j_1 \in O$ that conflicts with $i_1$. (proof later)

- Form a new set $O'$ by removing $j_1$ from $O$ and adding $i_1$, that is $O' = (O - \{j_1\}) \cup \{i_1\}$.
- From claim, $O'$ is a *feasible* solution (no conflicts).
- Since $|O'| = |O|$, $O'$ is also an optimum solution and it contains $i_1$. $\square$

# Proof of Claim

**Claim**

*If $i_1 \notin O$ then there is exactly one interval $j_1 \in O$ that conflicts with $i_1$.*

**Proof.**

- Suppose $j_1, j_2 \in O$ such that $j_1 \neq j_2$ and both $j_1$ and $j_2$ conflict with $i_1$.
- Since $i_1$ has earliest finish time, $j_1$ and $i_1$ overlap at $f(i_1)$.
- For same reason $j_2$ also overlaps with $i_1$ at $f(i_1)$.
- Implies that $j_1, j_2$ overlap at $f(i_1)$ contradicting the feasibility of $O$.

See figure in next slide. $\square$

# Figure for proof of Claim

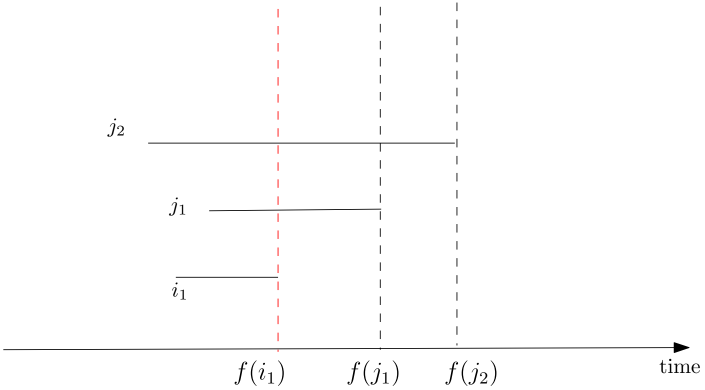


Figure: Since $i_1$ has the earliest finish time, any interval that conflicts with it does so at $f(i_1)$. This implies $j_1$ and $j_2$ conflict.

# Proof of Optimality of Earliest Finish Time First

## Proof by Induction on number of intervals.

**Base Case:** $n = 1$. Trivial since Greedy picks one interval.
**Induction Step:** Assume theorem holds for $i < n$.
Let $I$ be an instance with $n$ intervals
$I'$: $I$ with $i_1$ and all intervals that overlap with $i_1$ removed
$G(I), G(I')$: Solution produced by Greedy on $I$ and $I'$
From Lemma, there is an optimum solution $O$ to $I$ and $i_1 \in O$.
Let $O' = O - \{i_1\}$. $O'$ is a solution to $I'$.

$$\begin{aligned} |G(I)| &= 1 + |G(I')| \quad \text{(from Greedy description)} \\ &\geq 1 + |O'| \quad \text{(By induction, } G(I') \text{ is optimum for } I'\text{)} \\ &= |O| \end{aligned}$$

$\square$

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```

# Implementation and Running Time

```
Initially R is the set of all requests
A is empty (* A will store all the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R such that finishing time of i is least
    if i does not overlap with requests in A
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- Keep track of the finishing time of the last request added to $A$. Then check if starting time of $i$ later than that
- Thus, checking non-overlapping is $O(1)$
- Total time $O(n \log n + n) = O(n \log n)$

# Comments

- Interesting Exercise: smallest interval first picks at least half the optimum number of intervals.
- Instead of maximizing the total number of requests, associate *value/weight* with each job that is scheduled. Try to schedule jobs to maximize total value/weight. No greedy algorithm. Will be seen later in this course to illustrate dynamic programming.
- All requests need not be known at the beginning. Such *online* algorithms are a subject of research

# Scheduling all Requests

**Input** A set of lectures, with start and end times

**Goal** Find the minimum number of classrooms, needed to schedule all the lectures such two lectures do not occur at the same time in the same room.

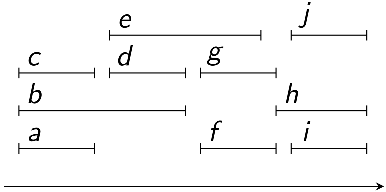


Figure: A schedule requiring 4 classrooms

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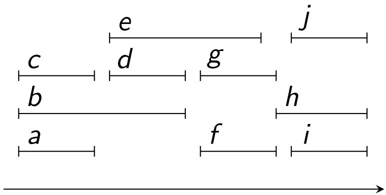


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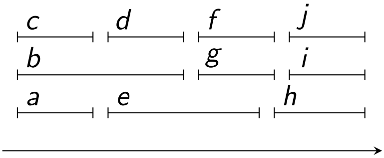


Figure: A schedule requiring 3 classrooms

# Greedy Algorithm

```
Initially R is the set of all requests
d = 0 (* number of classrooms *)
while R is not empty
    choose i ∈ R
    if i can be scheduled in some class-room k ≤ d
        schedule lecture i in class-room k
    else
        allocate a new class-room d+1 and schedule lecture i in d+1
        d = d+1
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What order should we process requests in?

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```

What order should we process requests in? According to start times (breaking ties arbitrarily)

# Depth of Lectures

## Definition

- For a set of lectures $R$, $k$ are said to be in conflict if there is some time $t$ such that there are $k$ lectures going on at time $t$.

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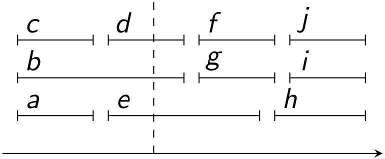

# Depth and Number of Class-rooms

**Lemma**

*For any set $R$ of lectures, the number of class-rooms required is at least the depth of $R$.*

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**Proof.**

All lectures that are in conflict must be scheduled in different rooms. $\square$

# Number of Class-rooms used by Greedy Algorithm

## Lemma

*Let $d$ be the depth of the set of lectures $R$. The number of class-rooms used by the greedy algorithm is $d$.*

## Proof.

- Suppose the greedy algorithm uses more that $d$ rooms. Let $j$ be the first lecture that is scheduled in room $d + 1$.

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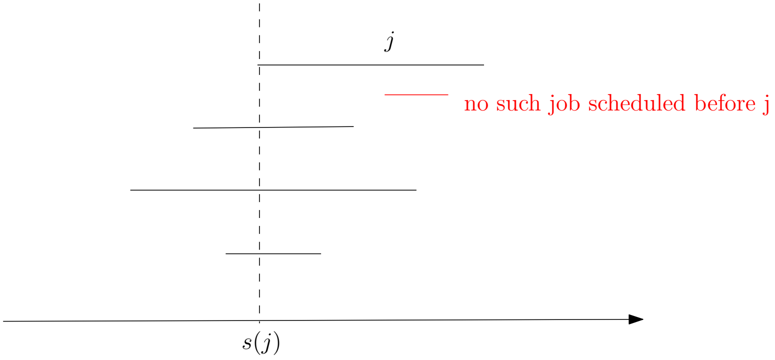
- Since we process lectures according to start times, there are $d$ lectures that start (at or) before $j$ and which are in conflict with $j$.
- Thus, *at the start time of $j$*, there are at least $d + 1$ lectures in conflict, which contradicts the fact that the depth is $d$. □

# Figure



# Correctness

**Observation**

*The greedy algorithm does not schedule two overlapping lectures in the same room.*

**Theorem**

*The greedy algorithm is correct and uses the optimal number of class-rooms.*

# Implementation and Running Time

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- Checking conflict takes $O(d)$ time.
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- Pre-sort according to start times. Picking lecture with earliest start time can be done in $O(1)$ time.
- Keep track of the finish time of last lecture in each room.
- With priority queues, checking conflict takes $O(\log d)$ time.
- Total time $= O(n \log n + n \log d) = O(n \log n)$

# Scheduling to Minimize Lateness

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job $i$ starts at time $s_i$ then it will finish at time $f_i = s_i + t_i$, where $t_i$ is its processing time.
- The lateness of a job is $l_i = \max(0, f_i - d_i)$.
- Schedule all jobs such that $L = \max l_i$ is minimized.

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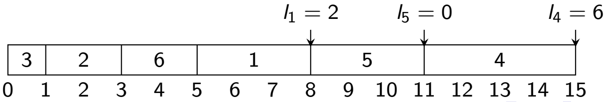
| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $t_i$ | 3 | 2 | 1 | 4 | 3 | 2 |
| $d_i$ | 6 | 8 | 9 | 9 | 14 | 15 |



# A Simpler Feasibility Problem

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- If a job $i$ starts at time $s_i$ then it will finish at time $f_i = s_i + t_i$, where $t_i$ is its processing time.
- Schedule all jobs such that each of them completes before its deadline (in other words $L = \max_i l_i = 0$).

**Definition**

A schedule is feasible if all jobs finish before their deadline.

# Greedy Template

```
Initially R is the set of all requests
curr-time = 0
while R is not empty
    choose i ∈ R
    curr-time = curr-time + t_i
    if (curr-time > d_i) then
        return ‘‘no feasible schedule’’
end while
return ‘‘found feasible schedule’’
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```

Main task: Decide the order in which to process jobs in $R$

# Three Algorithms

- Shortest job first — sort according to $t_i$.
- Shortest slack first — sort according to $d_i - t_i$.
- Earliest deadline first — sort according to $d_i$.

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Counter examples for first two: exercise

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**Theorem**

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Proof via an exchange argument.

Idle time: time during which machine is not working.

**Lemma**

*If there is a feasible schedule then there is one with no idle time before all jobs are finished.*

# Inversions

## Definition

A schedule $S$ is said to have an inversion if there are jobs $i$ and $j$ such that $S$ schedules $i$ before $j$, but $d_i > d_j$.

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A schedule $S$ is said to have an inversion if there are jobs $i$ and $j$ such that $S$ schedules $i$ before $j$, but $d_i > d_j$.

## Claim

*If a schedule $S$ has an inversion then there is an inversion between two* adjacently *scheduled jobs.*

Proof: exercise.

# Main Lemma

## Lemma

*If there is a feasible schedule, then there is one with no inversions.*

## Proof Sketch.

Let $S$ be a schedule with minimum number of inversions.

- If $S$ has 0 inversions, done.
- Suppose $S$ has one or more inversions. By claim there are two adjacent jobs $i$ and $j$ that define an inversion.
- Swap positions of $i$ and $j$.
- New schedule is still feasible. (Why?)
- New schedule has one fewer inversion — contradiction!

□

# Back to Minimizing Lateness

**Goal:** schedule to minimize $L = \max_i l_i$.

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Given a lateness bound $L$, can we check if there is a schedule such that $\max_i l_i \leq L$?

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Yes! Set $d'_i = d_i + L$ for each job $i$. Use feasibility algorithm with new deadlines.

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Yes! Set $d_i' = d_i + L$ for each job $i$. Use feasibility algorithm with new deadlines.

How can we find *minimum* $L$? Binary search!

# Binary search for finding minimum lateness

$L = L_{\min} = 0$
$L_{\max} = \sum_i t_i$ // *why is this sufficient?*
While $L_{\min} < L_{\max}$ do
    $L = \lfloor (L_{\max} + L_{\min})/2 \rfloor$
    check if there is a feasible schedule with lateness $L$
    if ''yes'' then $L_{\max} = L$
    else $L_{\min} = L + 1$
endwhile
return $L$

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Running time: $O(n \log n \cdot \log T)$ where $T = \sum_i t_i$

- $O(n \log n)$ for feasibility test (sort by deadlines)
- $O(\log T)$ calls to feasibility test in binary search

# Do we need binary search?

What happens in each call?
EDF algorithm with deadlines $d'_i = d_i + L$.

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Greedy with EDF schedules the jobs in the same order for all $L$!!!

Maybe there is a direct greedy algorithm for minimizing maximum lateness?

# Greedy Algorithm for Minimizing Lateness

```
Initially R is the set of all requests
curr-time = 0
curr-late = 0
while R is not empty
    choose i ∈ R with earliest deadline
    curr-time = curr-time + t_i
    late = curr-time - d_i
    curr-late = max (late, curr-late)
return curr-late
```

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Exercise: argue directly that above algorithm is correct (see book).

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Can be easily implemented in $O(n \log n)$ time after sorting jobs.

# Greedy Analysis: Overview

- Greedy's first step leads to an optimum solution. Show that there is an optimum solution leading from the first step of Greedy and then use induction. Example, Interval Scheduling.
- Greedy algorithm stays ahead. Show that after each step the solution of the greedy algorithm is at least as good as the solution of any other algorithm. Example, Interval scheduling.
- Structural property of solution. Observe some structural bound of every solution to the problem, and show that greedy algorithm achieves this bound. Example, Interval Partitioning.
- Exchange argument. Gradually transform any optimal solution to the one produced by the greedy algorithm, without hurting its optimality. Example, Minimizing lateness.

# Takeaway Points

- Greedy algorithms come naturally but often are incorrect. A proof of correctness is an absolute necessity.
- *Exchange* arguments are often the key proof ingredient. Focus on why the first step of the algorithm is correct: need to show that there is an optimum/correct solution with the first step of the algorithm.
- Thinking about correctness is also a good way to figure out which of the many greedy strategies is likely to work.