CS 473: Algorithms

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Part I

Dijkstra's Algorithm Recap

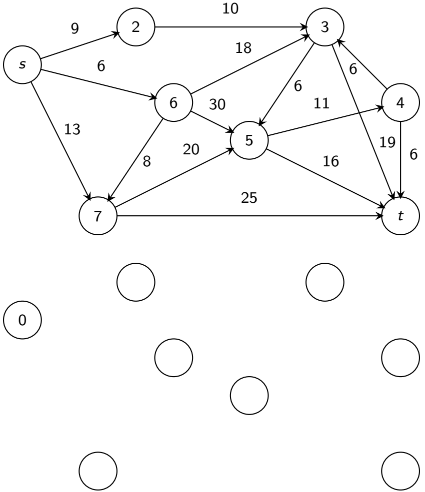
Dijkstra's Algorithm using Priority Queues

Shortest paths from node s to all nodes in V:

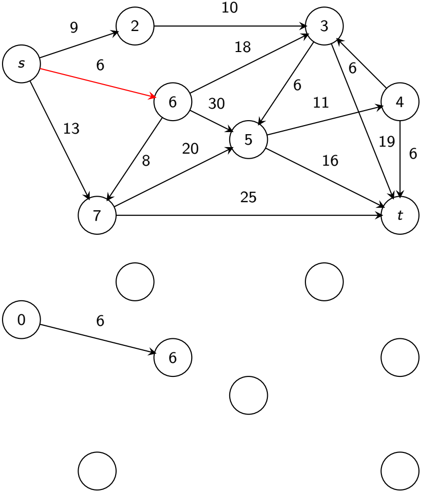
```
Q = makePQ()
insert(Q, (s,0))
for each node u ≠ s
    insert(Q, (u,∞))
S = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    S = S ∪ {v}
    For each u in Adj(v) do
        decreaseKey(Q, (u, min (dist(s,u), dist(s,v) + ℓ(v,u))))
```

Algorithm adds nodes to S in order of increasing distance from s

Example



Example



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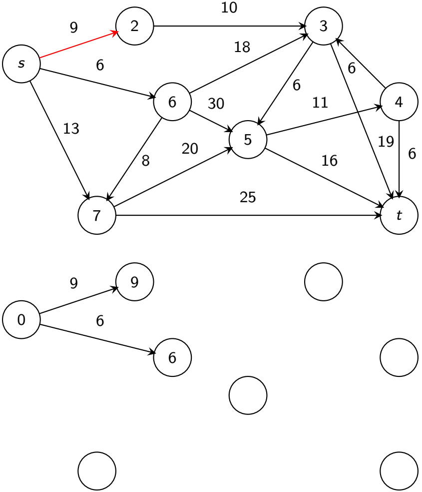

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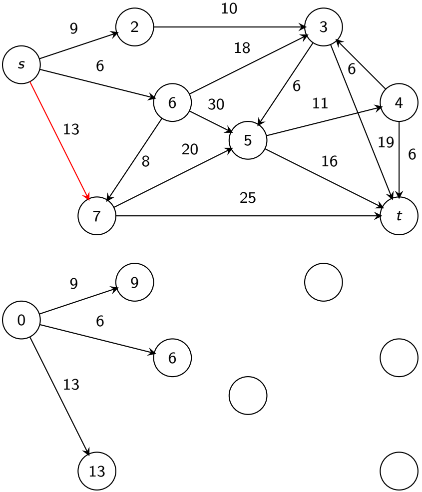

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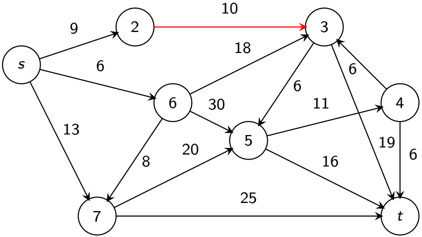

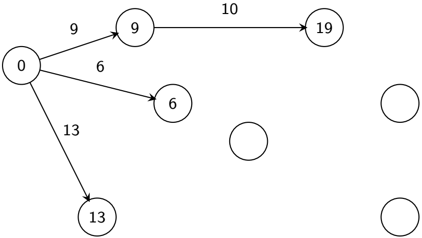

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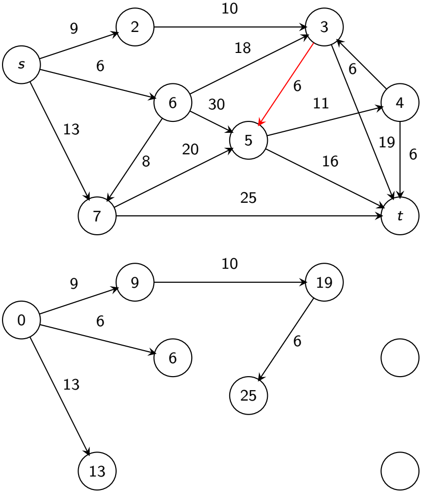

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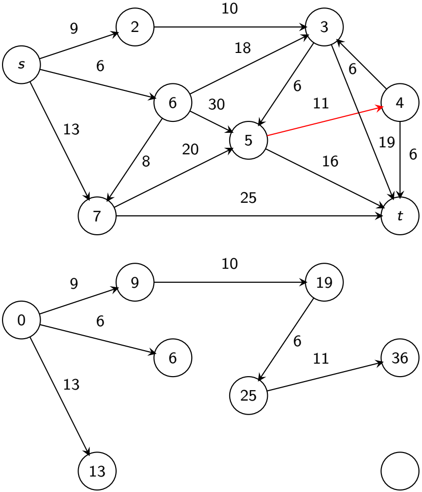

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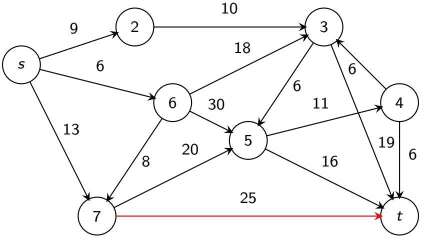
Example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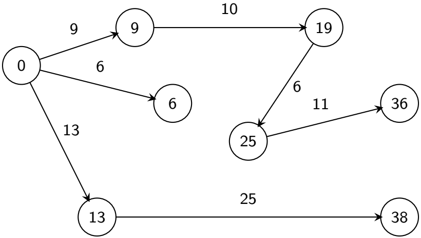
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Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to V.
Question: How do we find the paths themselves?

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Q = makePQ()
insert(Q, (s,0))
prev(s) = null
for each node u ≠ s
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S = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    S = S ∪ {v}
    For each u in Adj(v) do
        if (dist(s,v) + ℓ(v,u) < dist(s,u) ) then
            decreaseKey(Q, (u, dist(s,v) + ℓ(v,u)))
            prev(u) = v
```

Shortest Path Tree

Lemma

The edge set $(u, prev(u))$ is the reverse *of a shortest path tree rooted at s. For each u, the reverse of the path from u to s in the tree is a shortest path from s to u.*

Proof Sketch.

- The edgeset $\{(u, prev(u)) \mid u \in V\}$ induces a directed in-tree rooted at s (Why?)
- Use induction on $|S|$ to argue that the tree is a shortest path tree for nodes in V.

$\square$

Shortest paths to s

Dijkstra's algorithm gives shortest paths from s to all nodes in V.

How do we find shortest paths from all of V to s?

Shortest paths to s

Dijkstra's algorithm gives shortest paths from s to all nodes in V.

How do we find shortest paths from all of V to s?

- In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to disjtinguish.
- In directed graphs, use Dijkstra's algorithm in G^{rev}!

Part II

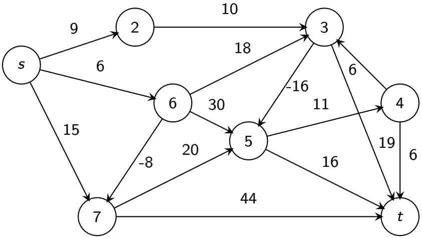
Shortest Paths with Negative Length Edges

Single-Source Shortest Paths with Negative Edge Lengths

Single-Source Shortest Path Problems

Input A directed graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.



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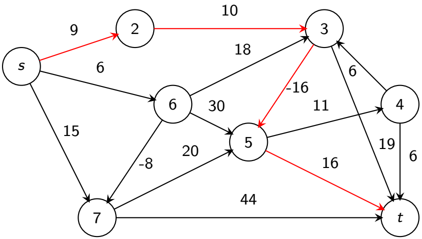
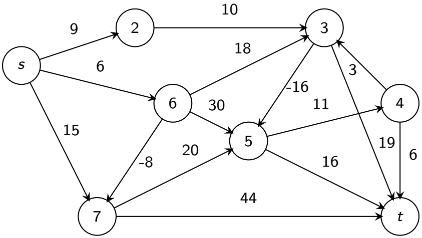


Negative Length Cycles

Definition

A cycle C is a negative length cycle if the sum of the edge lengths of C is negative.



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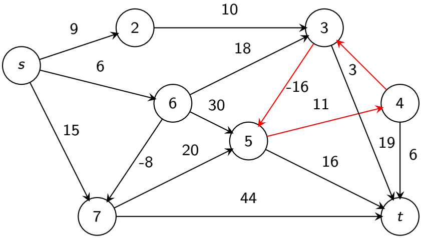

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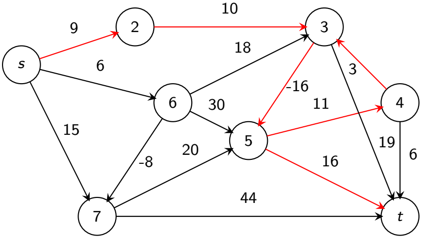

Shortest Paths and Negative Cycles

Given $G = (V, E)$ with edge lengths and s, t. Suppose

- G has a negative length cycle C, and
- s can reach C and C can reach t.

Question: What is the shortest *distance* from s to t?

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Lemma

If there is an efficient algorithm to find a shortest simple $s \to t$ path in a graph with negative edge lengths, then there is an efficient algorithm to find the longest *simple $s \to t$ path in a graph with positive edge lengths.*

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Finding the $s \to t$ longest path is difficult. NP-Hard!

Shortest Paths with Negative Edge Lengths: Problems

Algorithmic Problems

Input A directed graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t, either find a negative length cycle C that s can reach and t can reached from, or find a shortest path from s to t.
- Given node s, either find a negative length cycle C that s can reach or find shortest path distances from s to all other nodes.
- Check if G has a negative length cycle or not.

Why Negative Lengths?

Several Applications

- Shortest path problems useful in modeling many situations — in some negative lenths are natural
- Negative length cycle can be used to find arbitrage opportunities in currency trading
- Important sub-routine in algorithms for more general problem: minimum-cost flow

Application to Currency Trading

Currency Trading

Input n currencies and for each ordered pair (a, b) the *exchange rate* for converting one unit of a into one unit of b.

- Is there an arbitrage opportunity?
- Given currencies s, t what is the best way to convert s to t (perhaps via other intermediate currencies)?

Reducing Currency Trading to Shortest Paths

Observation: If we convert currency i to j via intermediate currencies $k_1, k_2, \ldots, k_h$ then one unit of i yields $exch(i, k_1) \times exch(k_1, k_2) \ldots \times exch(k_h, j)$ units of j.

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Create currency trading graph $G = (V, E)$:

- For each currency i there is a node $v_i \in V$
- $E = V \times V$: an edge for each pair of currencies
- edge length $\ell(v_i, v_j) =$

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Exercise: Verify that

- There is an arbitrage opportunity if and only if G has a negative length cycle.
- The best way to convert currency i to currency j is via a shortest path in G from i to j. If d is the distance from i to j then one unit of i can be converted into 2^d units of j.

Shortest Paths with Negative Edge Lengths: Problems

Algorithmic Problems

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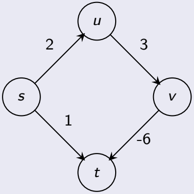
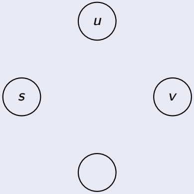
Dijkstra's Algorithm and Negative Lengths

Observation

With negative cost edges, Dijkstra's algorithm fails





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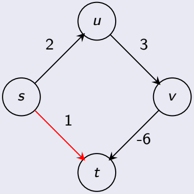

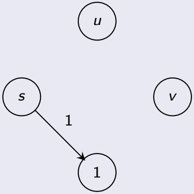

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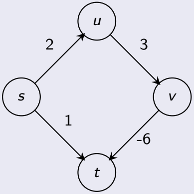

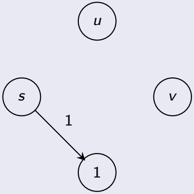

False assumption: Dijkstra's algorithm is based on the assumption that if $s = v_0 \rightarrow v_1 \rightarrow v_2 \ldots \rightarrow v_k$ is a shortest path from s to v_k then $dist(s, v_i) \leq dist(s, v_{i+1})$ for $0 \leq i < k$. Holds true only for non-negative edge lengths.

Shortest Paths with Negative Lengths

Lemma

Let G be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

- *$s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from s to v_i*

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- *$s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ is a shortest path from s to v_i*
- *False: $dist(s, v_i) \leq dist(s, v_k)$ for $1 \leq i < k$. Holds true only for non-negative edge lengths.*

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- *False: $dist(s, v_i) \leq dist(s, v_k)$ for $1 \leq i < k$. Holds true only for non-negative edge lengths.*

Cannot explore nodes in increasing order of distance! We need a more basic strategy.

A Generic Shortest Path Algorithm

- Start with distance estimate for each node $d(s, u)$ set to ∞
- Maintain the invariant that there is an $s \rightarrow u$ path of length $d(s, u)$. Hence $d(s, u) \geq dist(s, u)$.
- Iteratively refine $d(s, \cdot)$ values until they reach the correct value $dist(s, \cdot)$ values at termination

Question: How do we make progress?

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Given distance estimates $d(s, u)$ for each $u \in V$, an edge $e = (u, v)$ is tense if $d(s, v) > d(s, u) + \ell(u, v)$.

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```
Relax(e=(u,v))
    if (d(s,v) > d(s,u) + ℓ(u,v)) then
        d(s,v) = d(s,u) + ℓ(u,v)
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```

Proposition

Relax() maintains the invariant on $d(s, u)$ values.

A Generic Shortest Path Algorithm

```
d(s,s) = 0
for each node u ≠ s do
    d(s,u) = ∞

While there is a tense edge do
    Pick a tense edge e
    Relax(e)

Output d(s,u) values
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Lemma

If the algorithm terminates then $d(s, u) = dist(s, u)$ for each node u and s cannot reach a negative cycle.

Proof is left as an exercise after seeing future slides.

Dijkstra's Algorithm with Relax()

```
for each node u ≠ s
    d(s,u) = ∞
d(s,s) = 0
S = ∅
While (S ≠ V) do
    Let v be node in V − S with min d value
    S = S ∪ {v}
    For each edeg e in Adj(v) do
        Relax(e)
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Generic Algorithm: Ordering Relax operations

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Question: How do we pick edges to relax?

Observation: Suppose $s \to v_1 \to \ldots \to v_k$ is a shortest path.

If Relax(s, v_1), Relax(v_1, v_2), …, Relax(v_{k-1}, v_k) are done in *order* then $d(s, v_k) = dist(s, v_k)$!

Ordering Relax operations

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We don't know the shortest paths so how do we know the order to do the Relax operations?

Ordering Relax operations

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If Relax(s, v_1), Relax(v_1, v_2), ..., Relax(v_{k-1}, v_k) are done in *order* then $d(s, v_k) = dist(s, v_k)$! (Why?)

We don't know the shortest paths so how do we know the order to do the Relax operations?

We don't!

- Relax *all* edges (even those not tense) in some arbitrary order
- Iterate $|V| - 1$ times
- First iteration will do Relax(s, v_1) (and other edges), second round Relax(v_1, v_2) and in iteration k we do Relax(v_{k-1}, v_k).

Bellman-Ford Algorithm

```
for each u ∈ V
    d(s,u) = ∞
d(s,s) = 0

for i = 1 to |V| − 1 do
    for each edge e = (u, v) do
        Relax(e)

for each u ∈ V
    dist(s,u) = d(s,u)
```

Bellman-Ford Algorithm: Scanning Edges

One possible way to scan edges in each iteration.

```
Q is an empty queue
for each u ∈ V
    d(s,u) = ∞
    enqueue(Q, u)
d(s,s) = 0

for i = 1 to |V| − 1 do
    for i = 1 to |V| do
        u = dequeue(Q)
        for each edge e in Adj(u) do
            Relax(e)
        enqueue(Q,u)

for each u ∈ V
    dist(s,u) = d(s,u)
```

Example

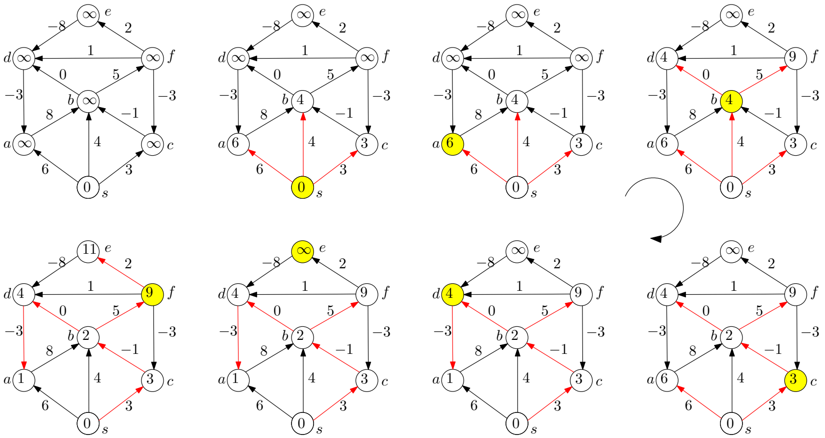

Figure: One iteration of Bellman-Ford that Relaxes all edges by processing nodes in the order s, a, b, c, d, e, f. Red edges indicate the prev pointers (in reverse)

Example

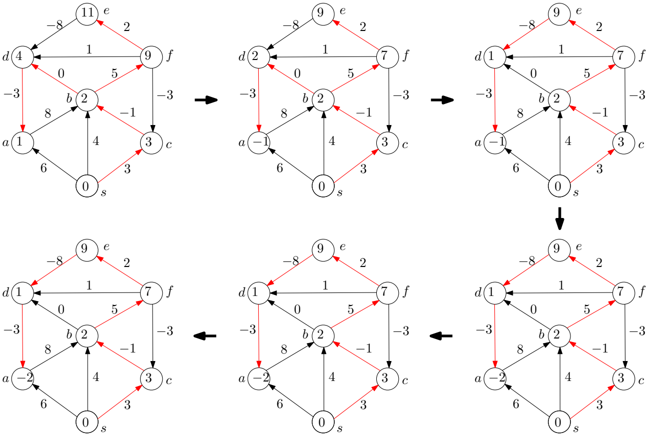

Figure: 6 iterations of Bellman-Ford starting with the first one from previous slidefigure. No changes in 5th iteration and 6th iteration.

Correctness of Bellman-Ford Algorithm

Lemma

Let G be a directed graph with arbitrary edge lengths. Let v be a node in V such that there is a shortest path in G from s to v with i hops/edges. Then, after i iterations of the for loop in the Bellman-Ford algorithm, $d(s, v) = dist(s, v)$

Proof.

By induction on i.

- Base case: $i = 0$. $d(s, s) = 0$ and $d(s, s) = dist(s, s)$.

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- Induction Step: Let $s \to v_1 \ldots \to v_{i-1} \to v$ be a shortest path from s to v of i hops.
    - v_{i-1} has a shortest path from s of $i - 1$ hops or less. (Why?). By induction, $d(s, v_{i-1}) = dist(s, v_{i-1})$ after $i - 1$ iterations.

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  - In iteration i, Relax(v_{i-1}, v_i) sets $d(s, v_i) = dist(s, v_i)$.

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    - In iteration i, Relax(v_{i-1}, v_i) sets $d(s, v_i) = dist(s, v_i)$.
    - Note: Relax does not change $d(s, u)$ once $d(s, u) = dist(s, u)$.

□

Correctness of Bellman-Ford Algorithm

Corollary

After $|V| - 1$ iterations of Bellman-Ford, $d(s, u) = dist(s, u)$ for any node u that has a shortest path from s.

Correctness of Bellman-Ford Algorithm

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After $|V| - 1$ iterations of Bellman-Ford, $d(s, u) = dist(s, u)$ for any node u that has a shortest path from s.

Note: If there is a negative cycle C such that s can reach C and C can reach s then $dist(s, u)$ is not defined.

Correctness of Bellman-Ford Algorithm

Corollary

After $|V| - 1$ iterations of Bellman-Ford, $d(s, u) = dist(s, u)$ for any node u that has a shortest path from s.

Note: If there is a negative cycle C such that s can reach C and C can reach s then $dist(s, u)$ is not defined.

Question: How do we know whether there is a negative cycle C reachable from s and whether $d(s, u)$ is valid or not?

Bellman-Ford to detect Negative Cycles

```
for each u ∈ V
    d(s,u) = ∞
d(s,s) = 0

for i = 1 to |V| − 1 do
    for each edge e = (u, v) do
        Relax(e)

for each u ∈ V
    dist(s,u) = d(s,u)

for each edge e = (u, v) do
    If e=(u,v) is tense then
        (* s can reach a negative length cycle C and C can reach v *)
        dist(s,v) = −∞
```

Correctness

Lemma

G has a negative cycle reachable from s if and only if there is a tense edge e after $|V| - 1$ iterations of Bellman-Ford.

Proof Sketch.

G has no negative length cycle reachable from s implies that all nodes u have a shortest path from s. Therefore $d(s, u) = dist(s, u)$ after the $|V| - 1$ iterations. Therefore, there cannot be any tense edges left.

Correctness

Lemma

G has a negative cycle reachable from s if and only if there is a tense edge e after $|V| - 1$ iterations of Bellman-Ford.

Proof Sketch.

G has no negative length cycle reachable from s implies that all nodes u have a shortest path from s. Therefore $d(s, u) = dist(s, u)$ after the $|V| - 1$ iterations. Therefore, there cannot be any tense edges left.

If there are no tense edges after $|V| - 1$ iterations, then $d(s, u) = dist(s, u)$ for all nodes. See exercise after the generic shortest path algorithm. $\square$

Finding the Paths

```
for each u ∈ V
    d(s,u) = ∞
    prev(u) = null
d(s,s) = 0
for i = 1 to |V| − 1 do
    for each edge e = (u, v) do
        Relax(e)
If there is a tense edge e then
    Output that s can reach a negative cycle C
Else
    for each u ∈ V
        dist(s,u) = d(s,u)
```

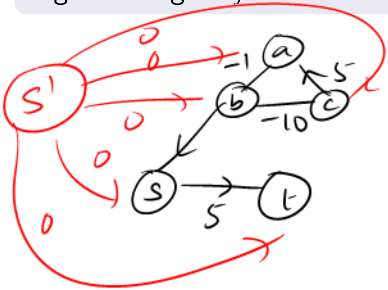
Modified Relax()

```
Relax(e=(u,v))
    if (d(s,v) > d(s,u) + ℓ(u, v)) then
        d(s,v) = d(s,u) + ℓ(u, v)
        prev(v) = u
```

Negative Cycle Detection

Negative Cycle Detection

Given directed graph G with arbirary edge lengths, does it have a negative length cycle?



Negative Cycle Detection

Negative Cycle Detection

Given directed graph G with arbirary edge lengths, does it have a negative length cycle?

- Bellman-Ford checks whether there is a negative cycle C that is reachable from a specific vertex s. There may negative cycles not reachable from s.
- Run Bellman-Ford $|V|$ times, once from each node u?

Negative Cycle Detection

- Add a new node s' and connect it to all nodes of G with zero length edges. Bellman-Ford from s' will fill find a negative length cycle if there is one. Exercise: why does this work?
- Negative cycle detection can be done with one Bellman-Ford invocation.

Running time for Bellman-Ford

- $O(|V|)$ iterations and $O(|E|)$ Relax() operations in each iteration. Each Relax() operation is $O(1)$ time.
- Total running time: $O(|V||E|)$.

Part III

Shortest Paths in DAGs

Shortest Paths in a DAG

Single-Source Shortest Path Problems

Input A directed acyclic graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.

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- Given node s find shortest path from s to all other nodes.

Simplification of algorithms for DAGs

- No cycles and hence no negative length cycles! Hence can find shortest paths even for negative length edges
- Can order nodes using topological sort

Algorithm for DAGs

- Want to find shortest paths from s. Ignore nodes not reachable from s.
- Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of G

Algorithm for DAGs

- Want to find shortest paths from s. Ignore nodes not reachable from s.
- Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of G

Observation:

- shortest path from s to v_i cannot use any node from $v_{i+1}, \ldots, v_n$
- can find shortest paths in topological sort order.

Algorithm for DAGs

```
for i = 1 to n
    d(s,v_i) = ∞
d(s,s) = 0

for i = 1 to n do
    for each edge e in Adj(v_i) do
        Relax(e)

return d values
```

Correctness: induction on i and observation in previous slide.
Running time: $O(m+n)$ time algorithm! Works for negative edge lengths and hence can find *longest* paths in a DAG