CS 473: Algorithms

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Part I

Recurrences

Solving Recurrences

Two general methods:

- Recursion tree method: need to do sums
  - elementary methods, geometric series
  - integration
- Guess and Verify
  - guessing involves intuition, experience and trial & error
  - verification is via induction

Recurrence: Example I

- Consider $T(n) = 2T(n/2) + n/\log n$.

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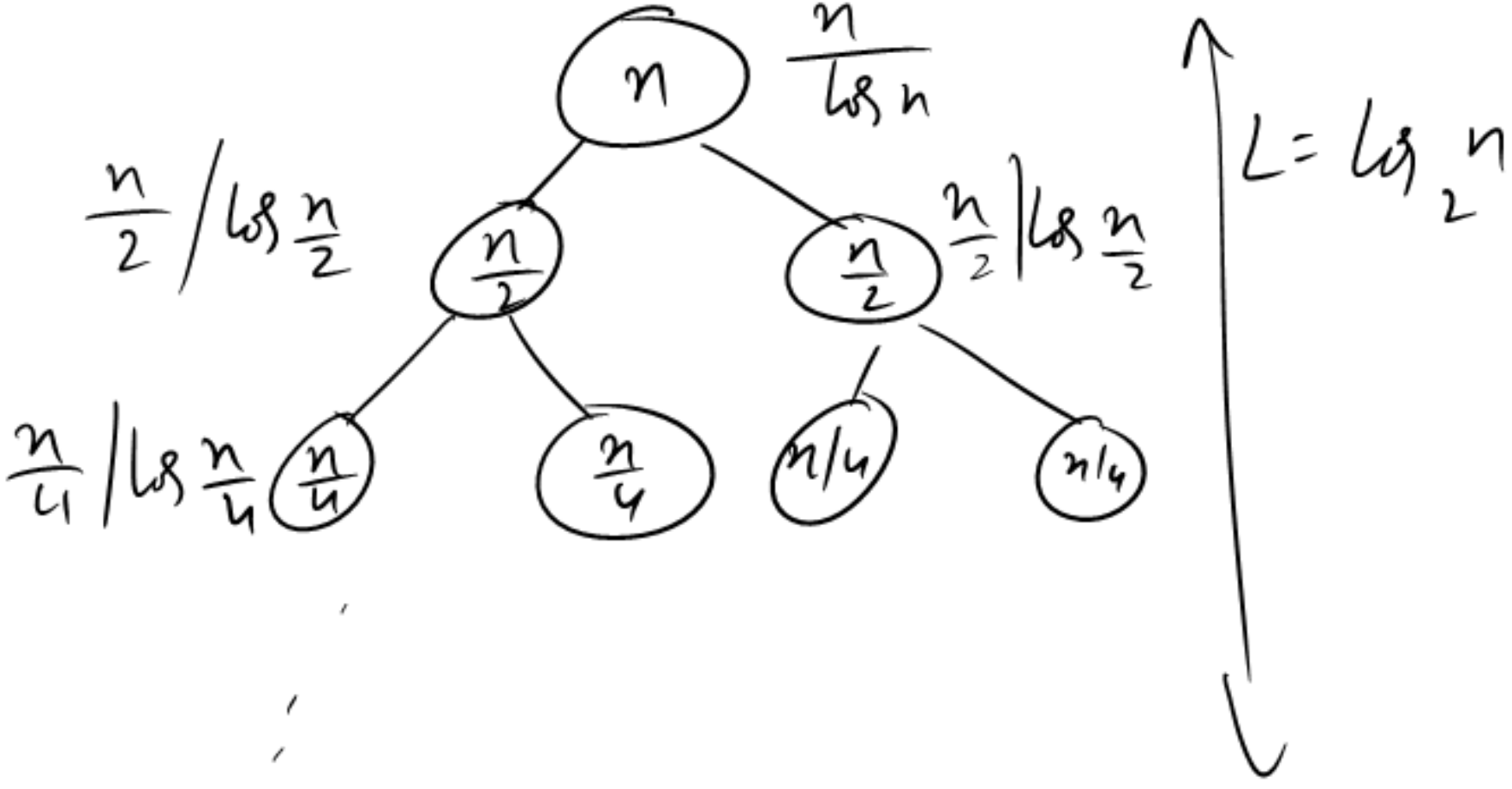
- Summing over all levels

$$\begin{aligned} T(n) &= \sum_{i=0}^{\log n - 1} 2^i \left[\frac{(n/2^i)}{\log(n/2^i)}\right] = \sum_{i=0}^{\log n - 1} \frac{n}{\log n - i} \\ &= n \sum_{j=1}^{\log n} \frac{1}{j} = nH_{\log n} = \Theta(n \log\log n) \end{aligned}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$$



Work at level $i = \frac{n}{\log \frac{n}{2^i}} = \frac{n}{\log n - i}$

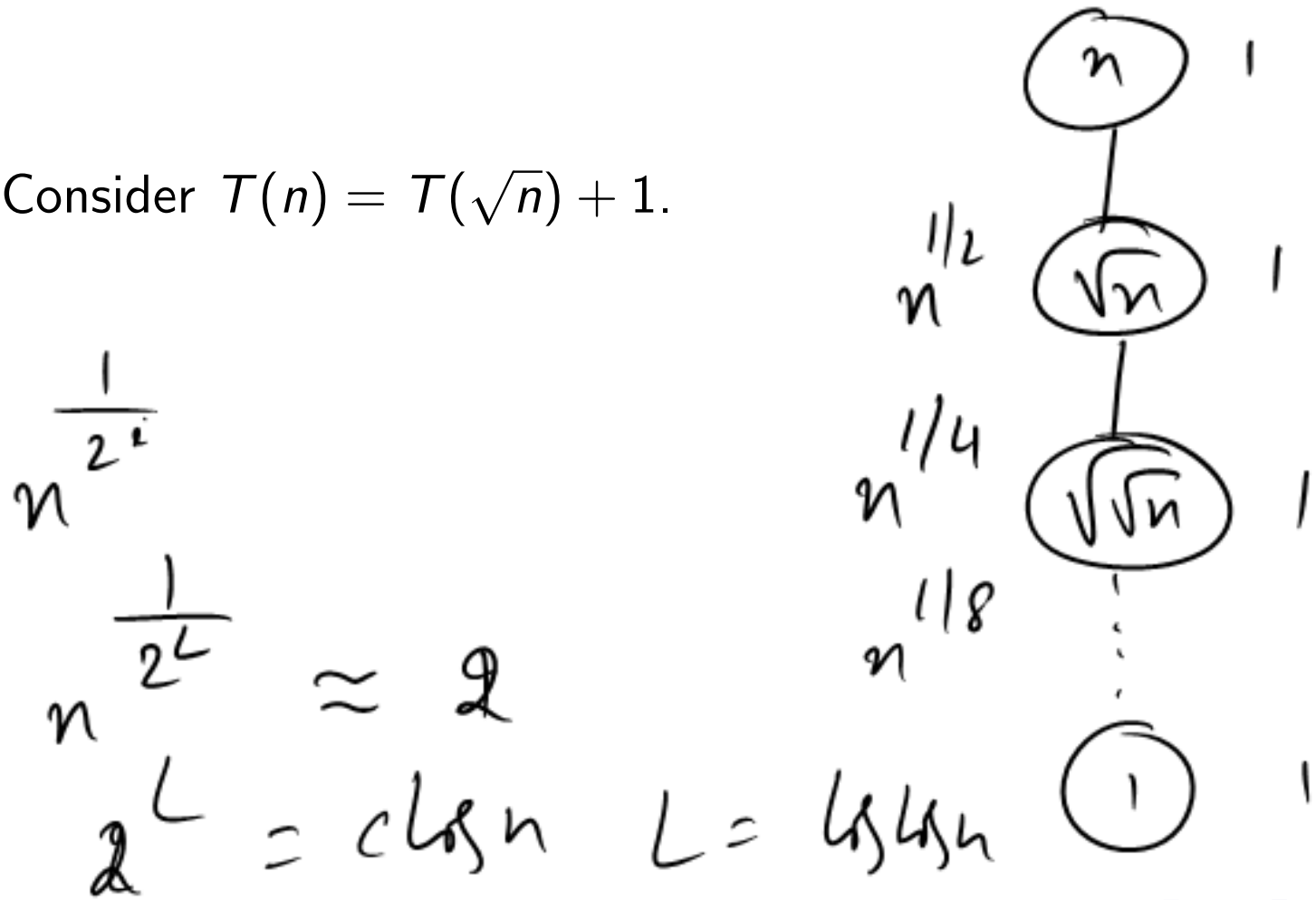
$$\frac{n}{\log n} + \frac{n}{\log n - 1} + \frac{n}{\log n - 2} + \cdots + \frac{n}{1}$$

$$= n\left(\frac{1}{\log n} + \frac{1}{\log n - 1} + \frac{1}{\log n - 2} + \cdots \to \frac{1}{1}\right)$$

$$= n H_{\log n} \simeq \Theta(n \log \log n)$$

Recurrence: Example II

- Consider $T(n) = T(\sqrt{n}) + 1$.



$$n^{\frac{1}{2^i}}$$

$$n^{\frac{1}{2^L}} \approx 2$$

$$2^L = c \log n \quad L = \log \log n$$

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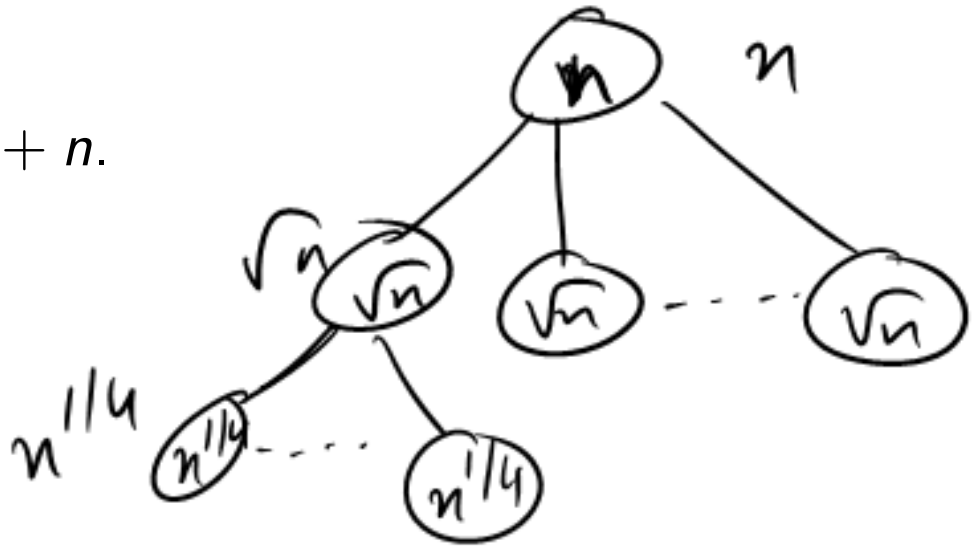
- Number of levels: $n^{2^{-L}} = 2$ means $L = \log \log n$
- Number of children at each level is 1, work at each node is 1
- Thus, $T(n) = \sum_{i=0}^{L} 1 = \Theta(L) = \Theta(\log \log n)$.

Recurrence: Example III

- Consider $T(n) = \sqrt{n}T(\sqrt{n}) + n$.



Work at level $i = n$

$L = \log \log n$ levels

So total $\Theta(n \log \log n)$

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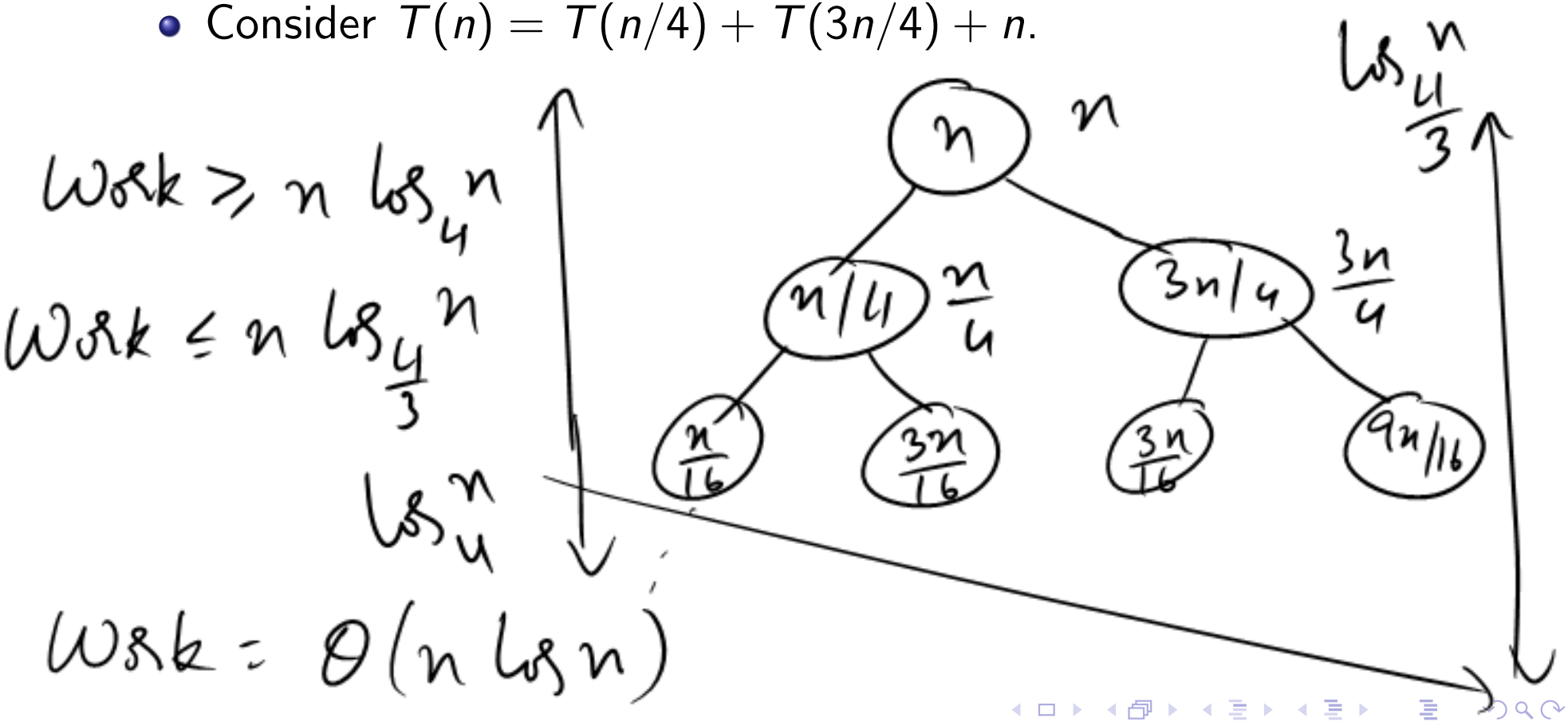
Recurrence: Example IV

- Consider $T(n) = T(n/4) + T(3n/4) + n$.



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- Total work in any level is at most n. Total work in any level without leaves is exactly n.
- Highest leaf is at level $\log_4 n$ and lowest leaf is at level $\log_{4/3} n$
- Thus, $n \log_4 n \leq T(n) \leq n \log_{4/3} n$, which means $T(n) = \Theta(n \log n)$

$$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{n}{2}\right) + n$$

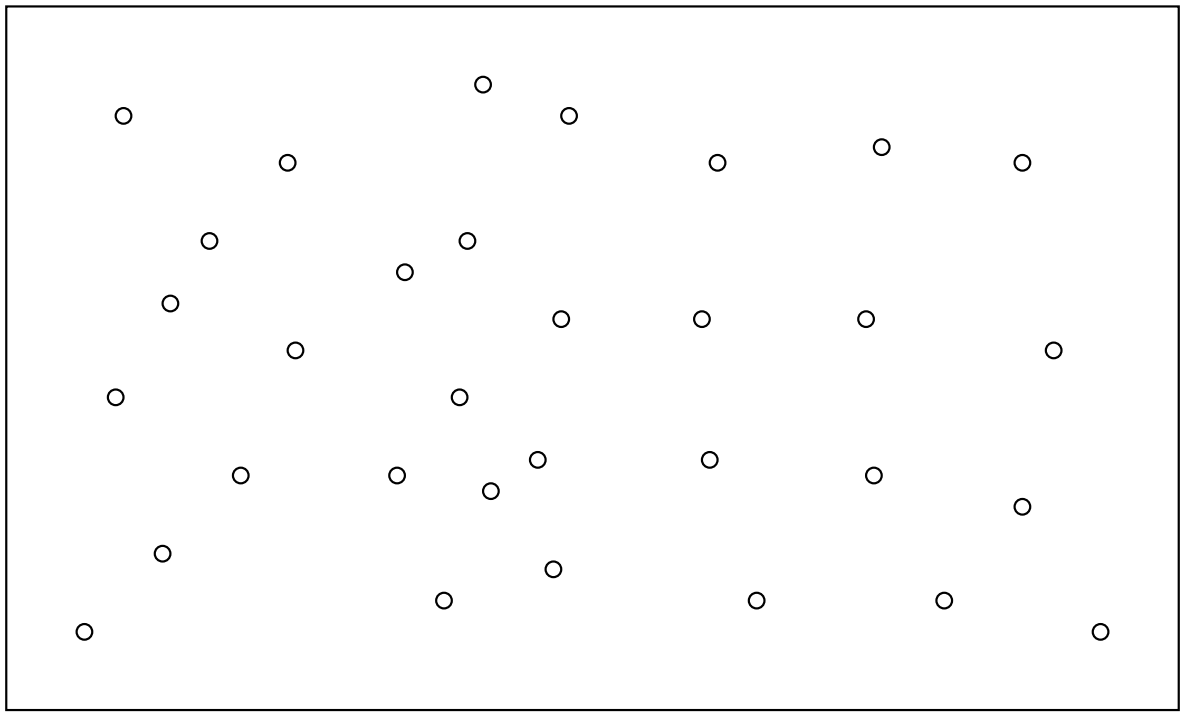
Part II

Closest Pair

Closest Pair

Input Given a set S of n points on the plane

Goal Find $p, q \in S$ such that $d(p, q)$ is minimum



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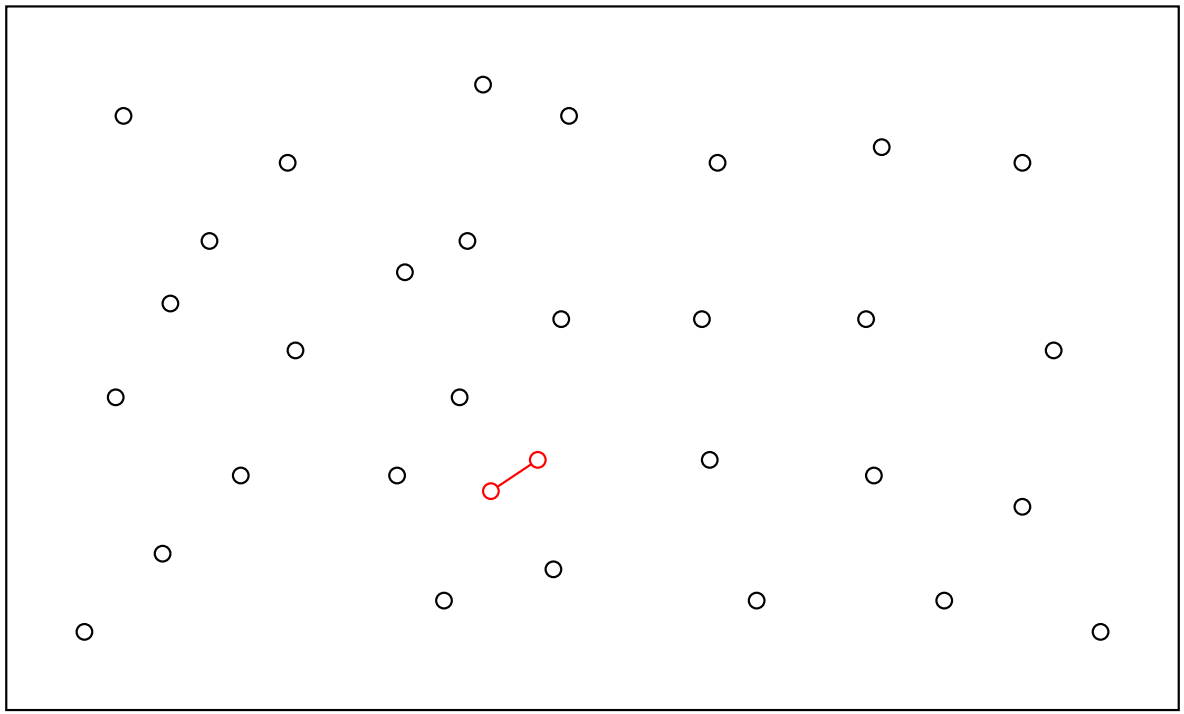

Applications

- Basic primitive used in graphics, vision, molecular modelling
- Ideas used in solving nearest neighbor, voronoi diagrams, euclidean MST

Algorithm

Algorithm

- Compute distance between every pair of points and find minimum
- Takes $O(n^2)$ time

Algorithm: Brute Force

- Compute distance between every pair of points and find minimum
- Takes $O(n^2)$ time
- Can we do better?

Closest Pair: 1-d case

Input Given a set S of n points on a line

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Algorithm

1. Sort points based on coordinate
2. Compute the distance between successive points, keeping track of the closest pair.

Running time $O(n \log n)$

Can we do this in better running time?
Can reduce Distinct Elements Problem (see lecture 1) to this problem in $O(n)$ time. Do you see how?

Generalizing 1-d case

Can we generalize 1-d algorithm to 2-d?

Sort according to x or y-coordinate??

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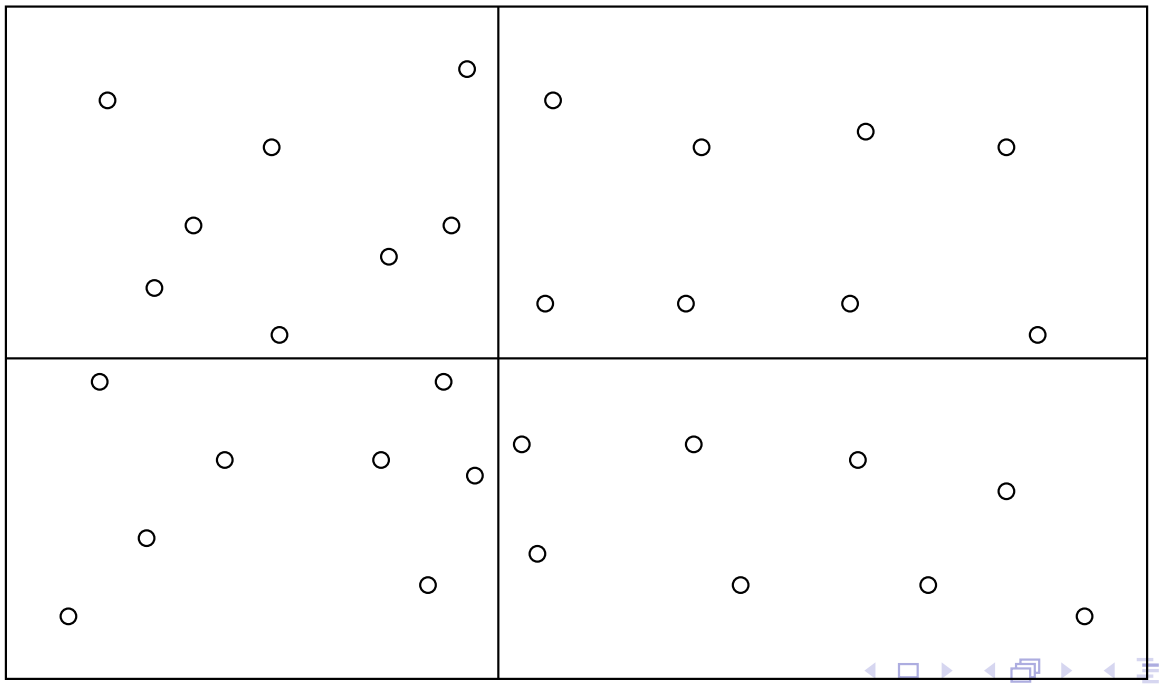
No easy generalization.

First Attempt

Divide and Conquer I

1. Partition into 4 quadrants of roughly equal size.
2. Find closest pair in each quadrant recursively
3. Combine solutions



First Attempt

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1. Partition into 4 quadrants of roughly equal size. Not always!
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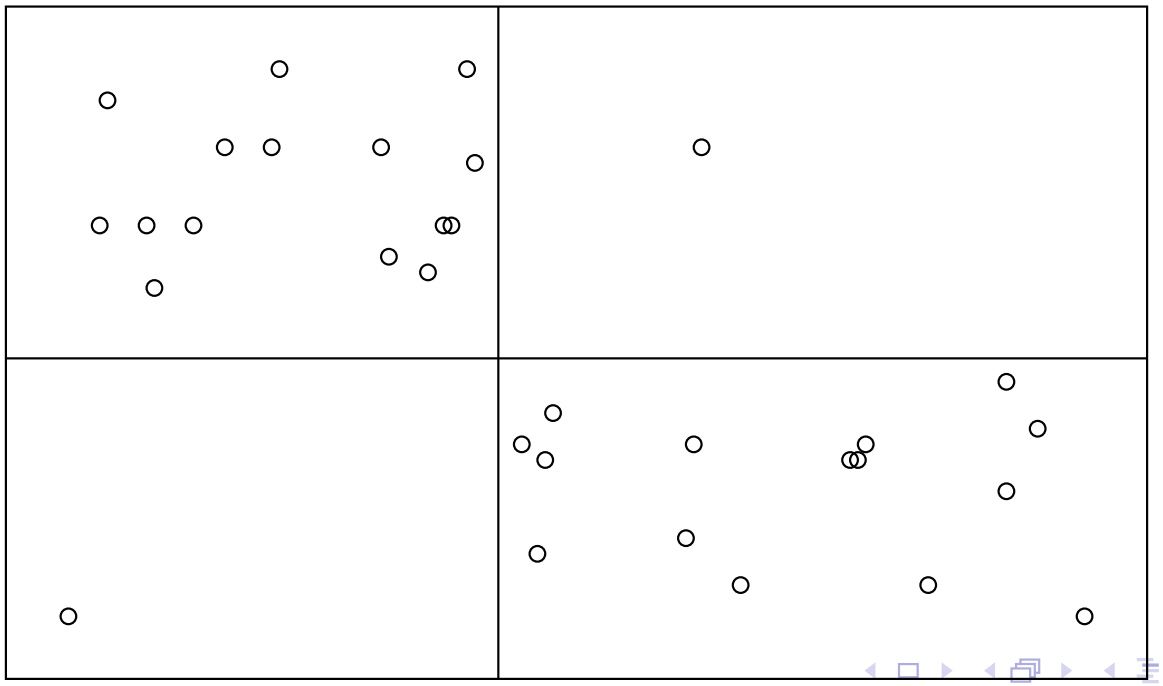
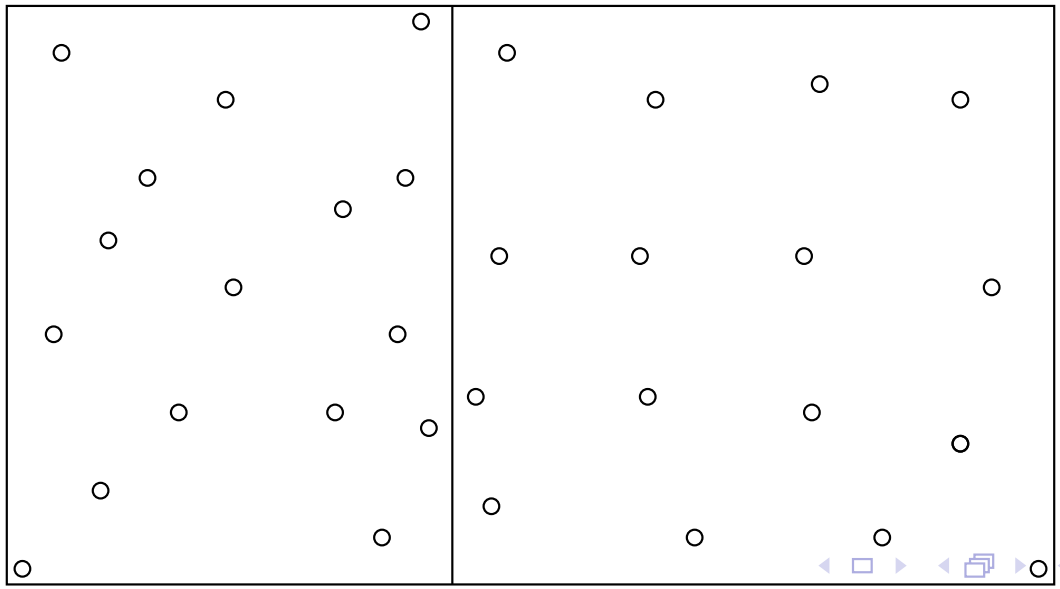


New Algorithm

Divide and Conquer II

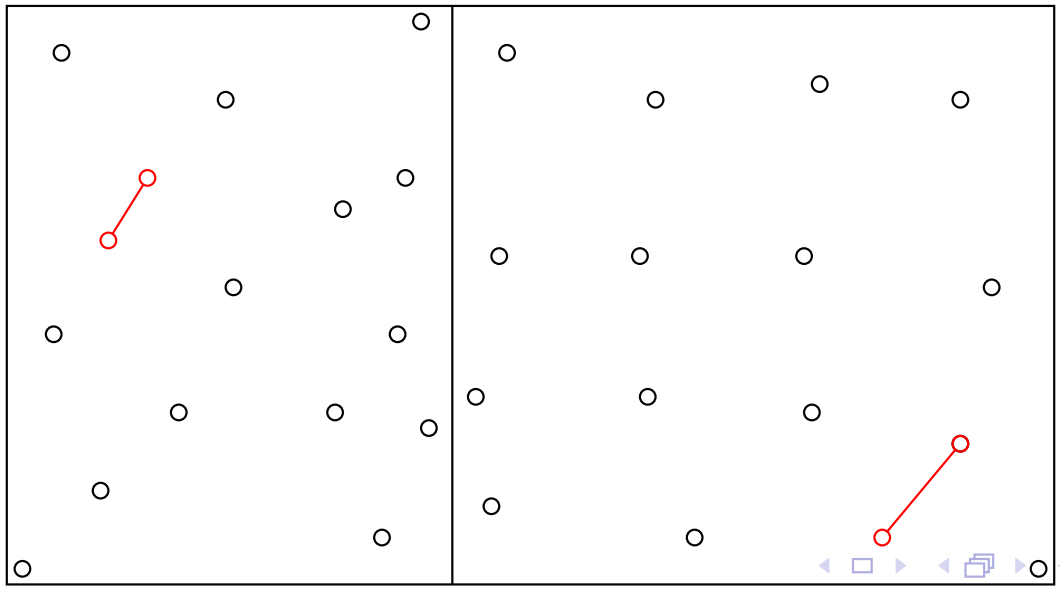
1. Divide the set of points into two equal parts via vertical line



New Algorithm

Divide and Conquer II

1. Divide the set of points into two equal parts via vertical line
2. Find closest pair in each half recursively



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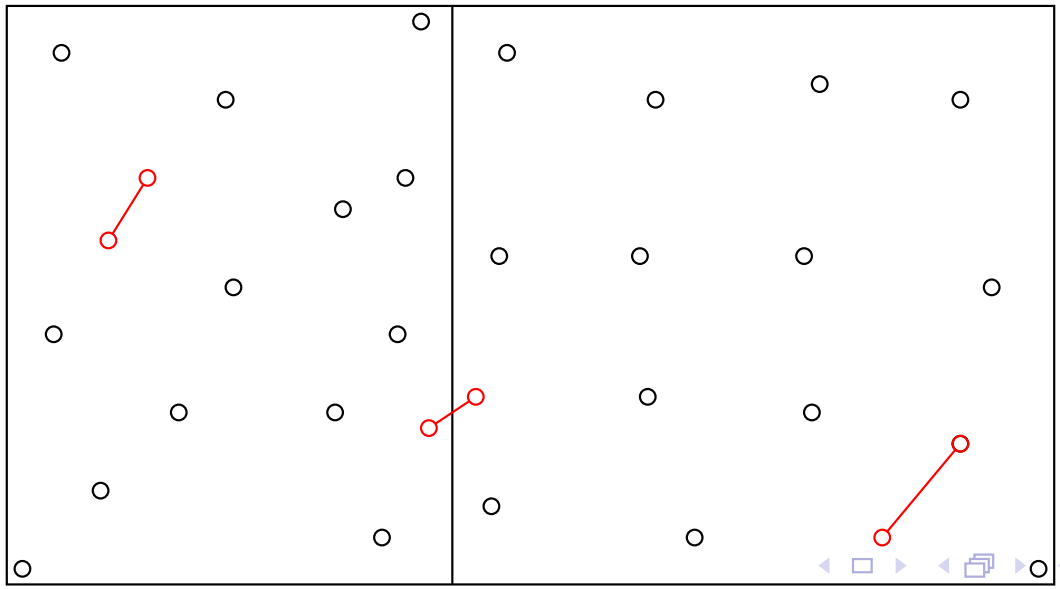

New Algorithm

Divide and Conquer II

1. Divide the set of points into two equal parts via vertical line
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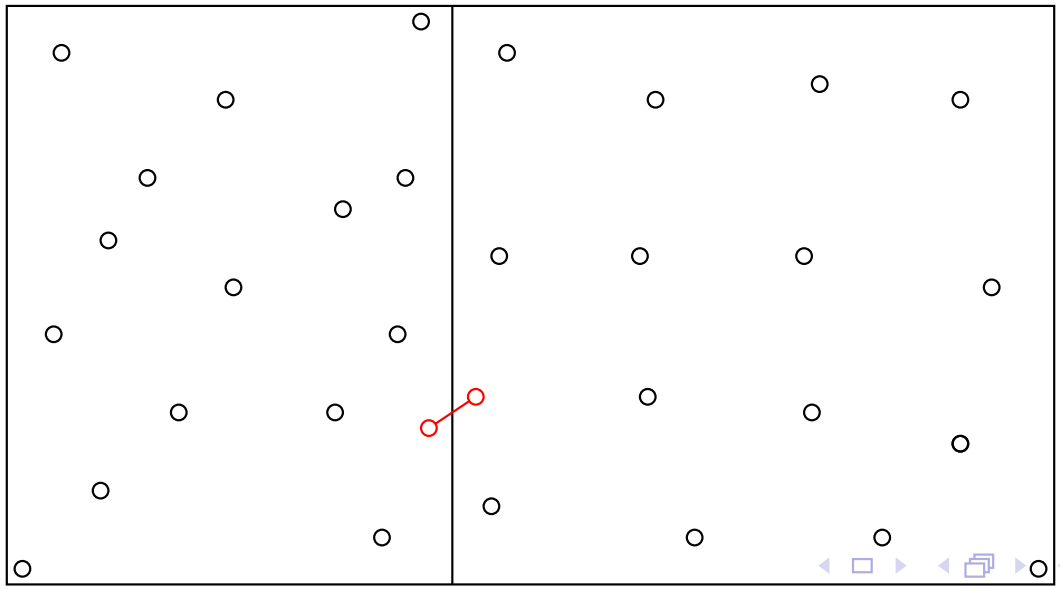
3. Find closest pair with one point in each half
4. Return the best pair among the above 3 solutions



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- Sort points based on x-coordinate and pick the median. Time $= O(n \log n)$

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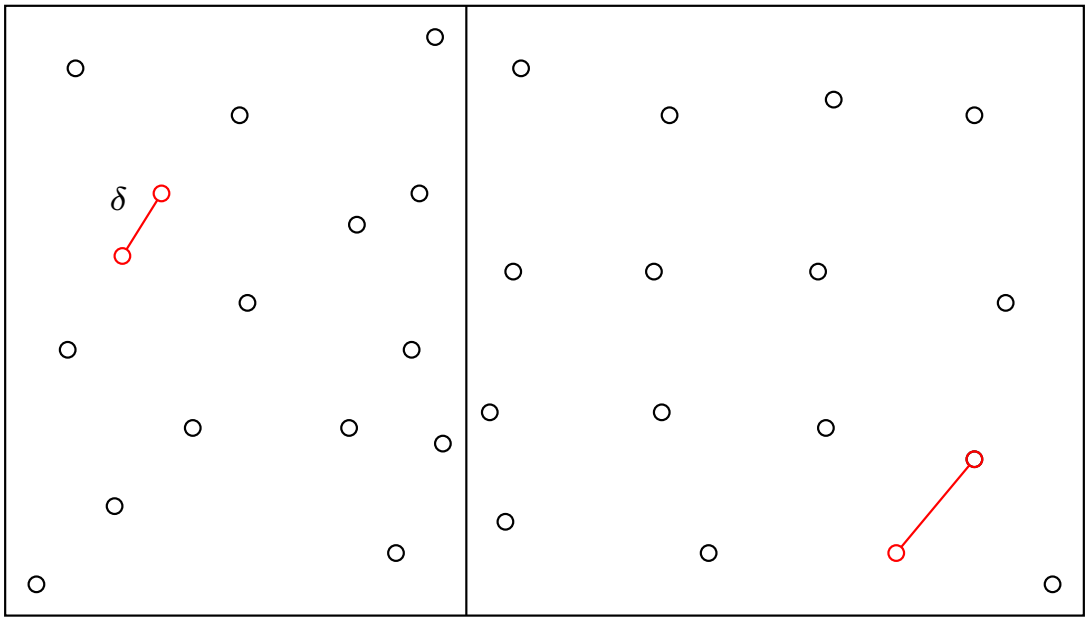
- Sort points based on x-coordinate and pick the median. Time $= O(n \log n)$
- How to find closest pair with points in different halves? $O(n^2)$ is trivial. Better?

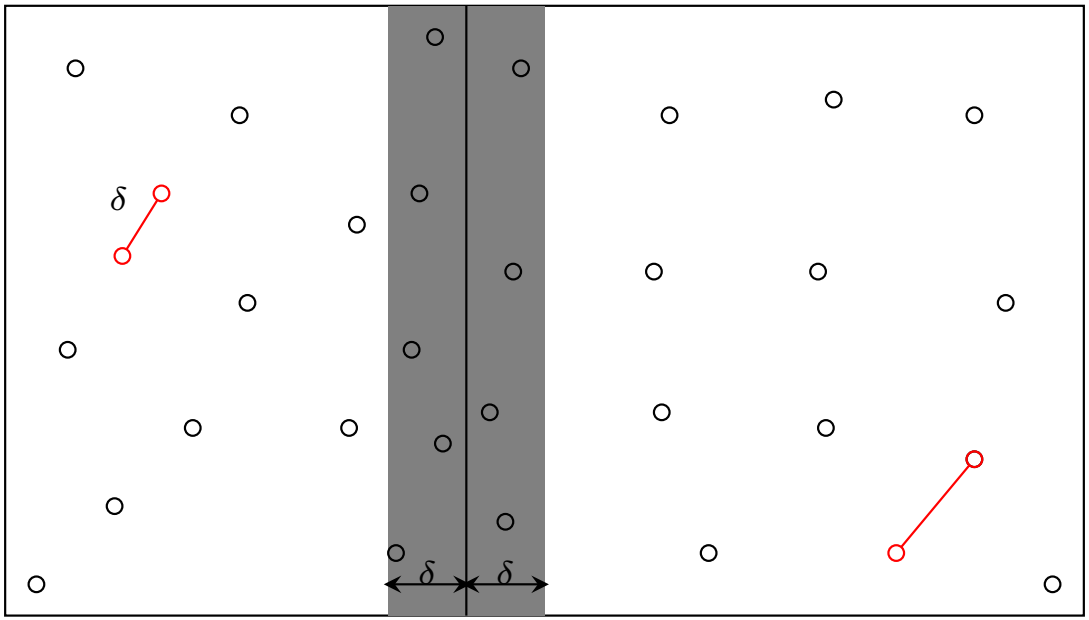
Combining Partial Solutions

- Does it take $O(n^2)$ to combine solutions?
- Let δ be the distance between closest pairs, where both points belong to the same half.

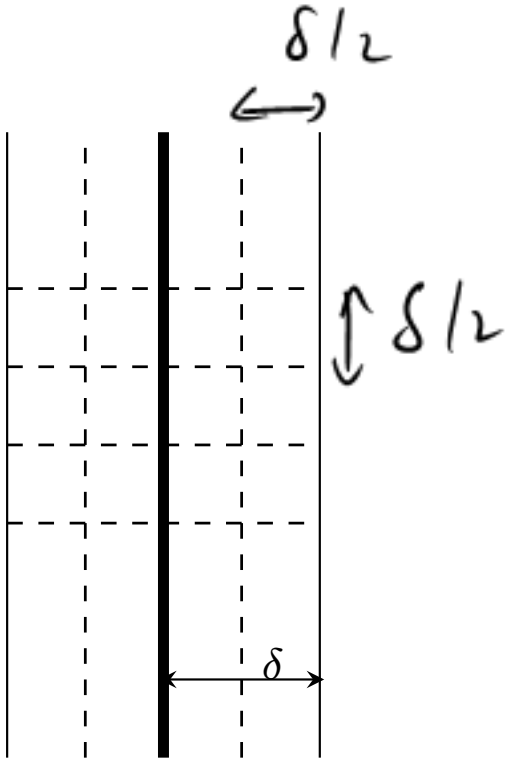


Combining Partial Solutions

- Let δ be the distance between closest pairs, where both points belong to the same half.
- Need to consider points within δ of dividing line

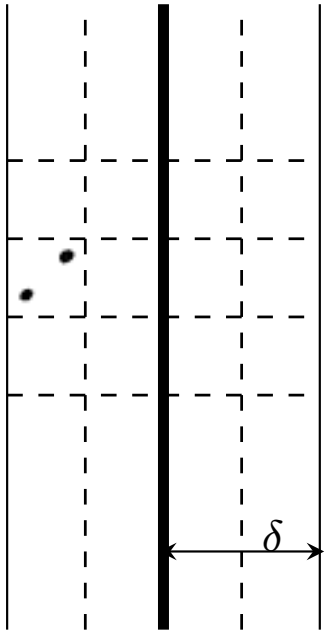


Sparsity of Band



Divide the band into square boxes of size $\delta/2$

Sparsity of Band



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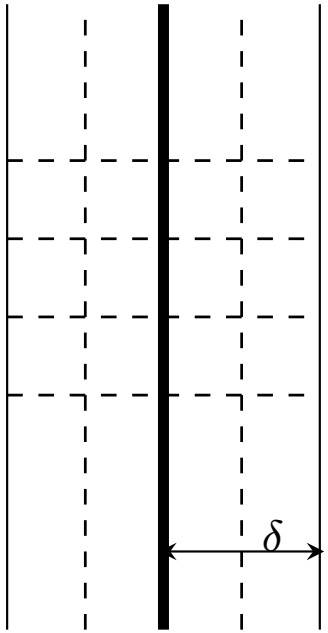

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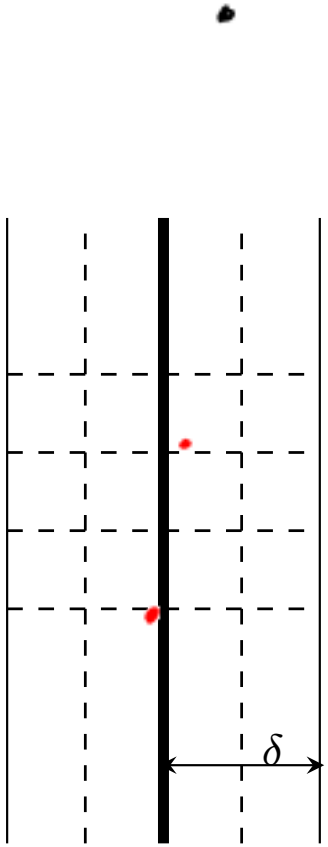
Proof.

If not, then there are a pair of points (both belonging to one half) that are at most $\sqrt{2}\delta/2 < \delta$ apart! □

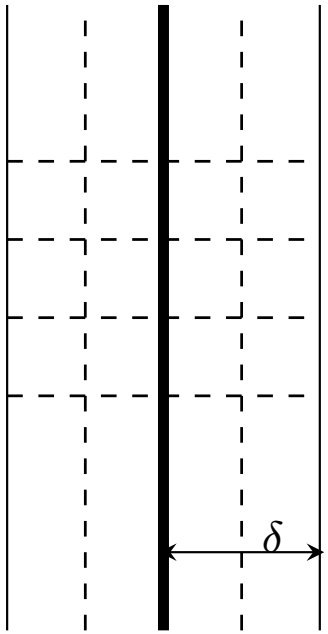
Searching within the Band



Lemma

Suppose a, b are at distance less than δ in the band. Then a, b have at most two rows of boxes between them.

Searching within the Band



Lemma

Suppose a, b are at distance less than δ in the band. Then a, b have at most two rows of boxes between them.

Proof.

Each row of boxes has height $\delta/2$. If more than two rows then distance between a, b greather than δ. □

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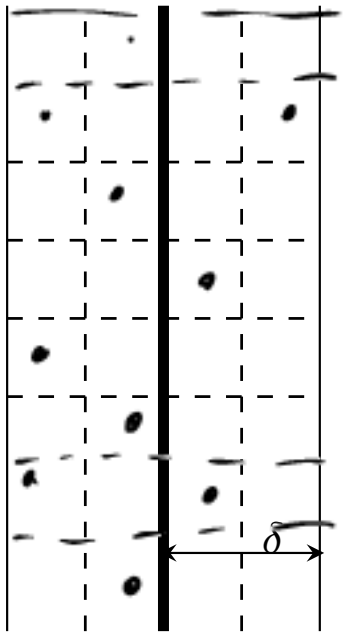
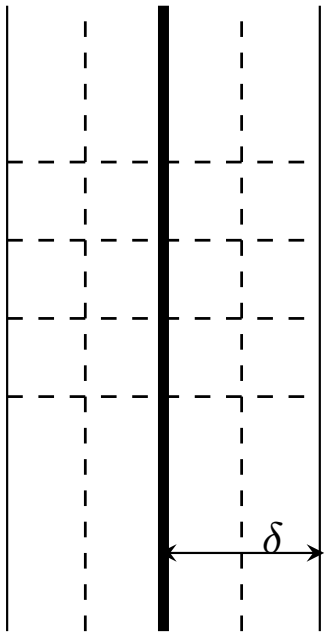


Corollary

Order points according to their y-coordinate. If p, q are such that $d(p, q) < \delta$ then p and q are within 12 positions in the sorted list.

Proof.

Searching within the Band



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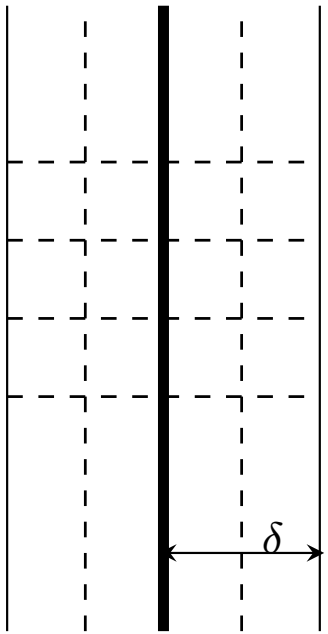

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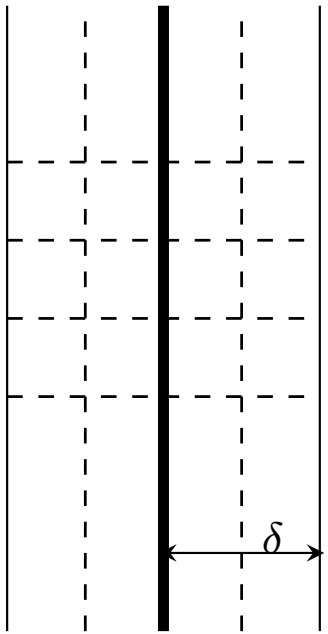

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- p and q are at least two rows apart in grid because each box has at most one point.
- $d(p, q) > 2(\delta/2) = \delta$! □

The Algorithm

1. Find vertical line L that splits the points into equal halves
2. Compute closest pair in the left half; let the distance be δ_1
3. Compute closest pair in right half; let the distance be δ_2
4. $\delta = \min(\delta_1, \delta_2)$
5. Delete points further than δ from L
6. Sort remaining points based on y-coordinate into an array A
7. for $i = 1$ to $|A| - 1$ do
   for $j = i + 1$ to $\min\{i + 11, |A|\}$ do
   If $(dist(A[i], A[j]) < \delta)$ update δ and closest pair

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- Step 5 takes $O(n)$ time

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- Step 1, involves sorting and scanning. Takes $O(n \log n)$ time.
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Improved Algorithm

Avoid repreated sorting of points in band: two options

- Sort all points by y-coordinate and store the list. In conquer step use this to avoid sorting
- Each recursive call returns a list of points sorted by their y-coordinates. Merge in conquer step in linear time.

Analysis: $T(n) \leq 2T(n/2) + O(n) = O(n \log n)$

Part III

Selecting in Unsorted Lists

Quick Sort

Quick Sort[Hoare]

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

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Example:

- array: 16, 12, 14, 20, 5, 3, 18, 19, 1
- pivot: 16
- split into 12, 14, 5, 3, 1 and 20, 19, 18 and recursively sort
- put them together with pivot in middle

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  - Theoretically, median can be found in linear time.
- Typically, pivot is the first or last element of array. Then,

$$T(n) = \max_{1 \leq k \leq n} (T(k-1) + T(n-k) + O(n))$$

In the worst case $T(n) = T(n-1) + O(n)$, which means $T(n) = O(n^2)$. Happens if array is already sorted and pivot is always first element.

Selection

Input Unsorted array A of n integers

Goal Find the j'th smallest number in A (*rank* j number)

Example

$A = \{4, 6, 2, 1, 5, 8, 7\}$ and $j = 4$. The jth smallest element is 5.

Median: $j = \lfloor n/2 \rfloor$

Algorithm I

1. Sort the elements in A
2. Pick jth element in sorted order

Time taken $= O(n \log n)$

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Do we need to sort? Is there an $O(n)$ time algorithm?

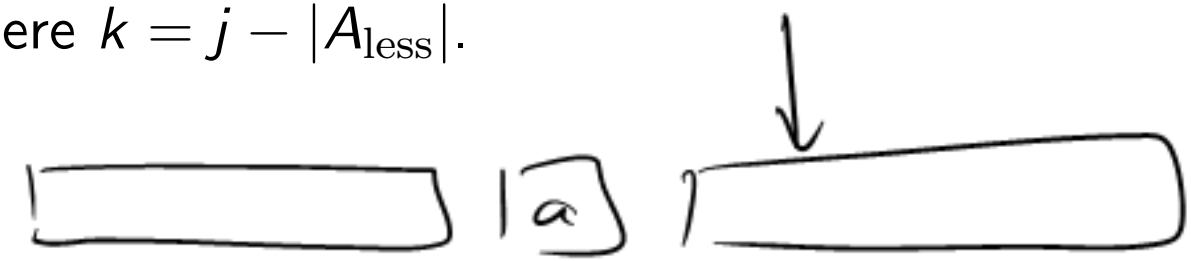
Algorithm II

If j is small or $n - j$ is small then

- Find j smallest/largest elements in A in $O(jn)$ time. (How?)
- Time to find median is $O(n^2)$.

Divide and Conquer Approach

1. Pick a pivot element a from A
2. Partition A based on a.
   $A_{\text{less}} = \{x \in A \mid x \leq a\}$ and $A_{\text{greater}} = \{x \in A \mid x > a\}$
3. $|A_{\text{less}}| = j$: return a
4. $|A_{\text{less}}| > j$: recursively find jth smallest element in A_{less}
5. $|A_{\text{less}}| < j$: recursively find kth smallest element in A_{greater}
   where $k = j - |A_{\text{less}}|$.



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Say A is sorted in increasing order and $j = n$.
Exercise: show that algorithm takes $\Omega(n^2)$ time

A Better Pivot

Suppose pivot is the ℓ'th smallest element where $n/4 \leq \ell \leq 3n/4$. That is pivot is *approximately* in the middle of A

Then $n/4 \leq |A_{\text{less}}| \leq 3n/4$ *and* $n/4 \leq |A_{\text{greater}}| \leq 3n/4$. If we apply recursion,

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Can we choose pivot deterministically?

Choosing the pivot

1. Partition array A into $\lceil n/5 \rceil$ lists of 5 items each.
   $L_1 = \{A[1], A[2], \ldots, A[5]\}$, $L_2 = \{A[6], \ldots, A[10]\}$, $\ldots$,
   $L_i = \{A[5i+1], \ldots, A[5i-4]\}$, $\ldots$,
   $L_{\lceil n/5 \rceil} = \{A[5\lceil n/5 \rceil - 4, \ldots, A[n]\}$.
2. For each i find median b_i of L_i using brute-force in $O(1)$ time.
   Total $O(n)$ time
3. Let $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
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Lemma

Median of B is an approximate *median of A. That is, if b is used a pivot to partition A, then $|A_{less}| \leq 7n/10 + 6$ and $|A_{greater}| \leq 7n/10 + 6$.*

Algorithm for Selection

```
SELECT(A, j):
    Form lists L_1, L_2, ..., L_⌈n/5⌉ where L_i = {A[5i − 4], ..., A[5i]}
    Find median b_i of each L_i using brute-force
    Find median b of B = {b_1, b_2, ..., b_⌈n/5⌉}
    Partition A into A_less and A_greater using b as pivot
    If (|A_less|) = j return b
    Else if (|A_less|) > j)
        return SELECT(A_less, j)
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How do we find median of B? Recursively!

Recursive algorithm for Selection

SELECT(A, j):
    Form lists $L_1, L_2, \ldots, L_{\lceil n/5 \rceil}$ where $L_i = \{A[5i-4], \ldots, A[5i]\}$
    Find median b_i of each L_i using brute-force
    B is the array of $b_1, b_2, \ldots, b_{\lceil n/5 \rceil}$.
    b = SELECT(B, $\lceil n/10 \rceil$)
    Partition A into A_{less} and A_{greater} using b as pivot
    If $(|A_{\text{less}}|) = j$ return b
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Exercise: show that $T(n) = O(n)$

Median of Medians: Proof of Lemma

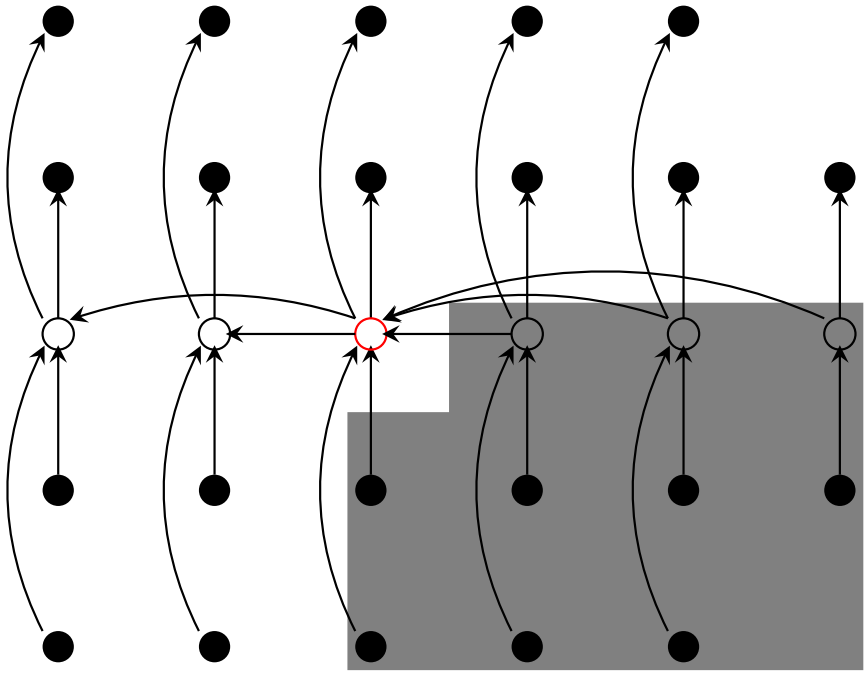

Figure: Shaded elements are all greater than b

Proposition

There are at least $3n/10 - 6$ elements greater than the median of medians b.

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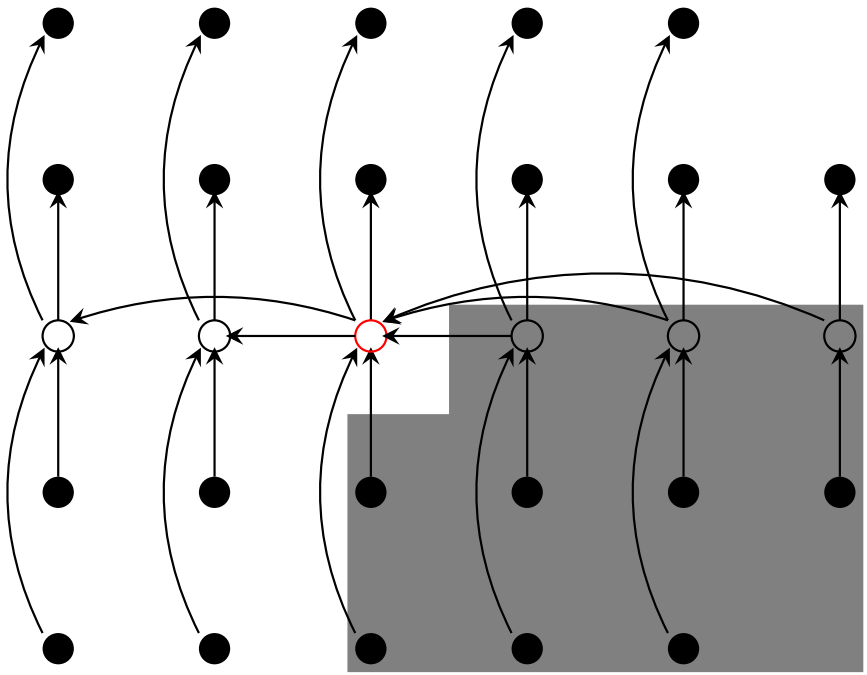

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Proof.

At least half of the $\lceil n/5 \rceil$ groups have at least 3 elements larger than b, except for last group and the group containing b. So b is less than

$$3(\lceil (1/2)\lceil n/5 \rceil \rceil - 2) \geq 3n/10 - 6 \quad \square$$

Median of Medians: Proof of Lemma

Proposition

There are at least $3n/10 - 6$ *elements greater than the median of medians* b.

Corollary

$|A_{less}| \leq 7n/10 + 6$.

Via symmetric argument,

Corollary

$|A_{greater}| \leq 7n/10 + 6$.

Questions to ponder

- Why did we choose lists of size 5? Will lists of size 3 work?
- Write a recurrence to analyze the algorithm's running time if we choose a list of size k.

Median of Medians Algorithm

Due to:

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All except Vaughn Pratt!