CS 473: Algorithms

Chandra Chekuri
chekuri@cs.illinois.edu
3228 Siebel Center

University of Illinois, Urbana-Champaign

Fall 2009

Part I

Knapsack

Knapsack Problem

Input Given a Knapsack of capacity W lbs. and n objects with ith object having weight w_i and value v_i; assume W, w_i, v_i are all positive integers

Goal Fill the Knapsack without exceeding weight limit while maximizing value.

Knapsack Problem

Input Given a Knapsack of capacity W lbs. and n objects with ith object having weight w_i and value v_i; assume W, w_i, v_i are all positive integers

Goal Fill the Knapsack without exceeding weight limit while maximizing value.

We saw that:

- Knapsack can be solved exactly in $O(nW)$ time via dynamic programming. Not polynomial time when W is large compared to n.
- Knapsack is NP-Complete

Knapsack Example

Example

Item	1	2	3	4	5
Value	1	6	18	22	28
Weight	1	2	5	6	7

If $W = 11$, the best is $\{3, 4\}$ giving value 40.

Greedy Approximation Algorithm

- Sort objects in decreasing order of v_i/w_i (bang per buck)

Greedy Approximation Algorithm

- Sort objects in decreasing order of v_i/w_i (bang per buck)
- Insert items in sorted order and item to knapsack if sufficient weight left.

Greedy Approximation Algorithm

- Sort objects in decreasing order of v_i/w_i (bang per buck)
- Insert items in sorted order and item to knapsack if sufficient weight left.
- Bad example: Two items: $v_1 = 1$, $w_1 = 1$, $v_2 = W - 1$, $w_2 = W$. Greedy will pack item 1 and stop and get value 1 while optimum solution is to pack item 2 of value $W - 1$.

Greedy Approximation Algorithm

- Sort objects in decreasing order of v_i/w_i (bang per buck)
- Insert items in sorted order and item to knapsack if sufficient weight left.
- Bad example: Two items: $v_1 = 1$, $w_1 = 1$, $v_2 = W - 1$, $w_2 = W$. Greedy will pack item 1 and stop and get value 1 while optimum solution is to pack item 2 of value $W - 1$.

Greedy Approximation Algorithm

- Sort objects in decreasing order of v_i/w_i (bang per buck)
- Insert items in sorted order and item to knapsack if sufficient weight left.
- Bad example: Two items: $v_1 = 1$, $w_1 = 1$, $v_2 = W - 1$, $w_2 = W$. Greedy will pack item 1 and stop and get value 1 while optimum solution is to pack item 2 of value $W - 1$.

Is Greedy really bad?

Greedy Approximation Algorithm

- Sort objects in decreasing order of v_i/w_i (bang per buck)
- Insert items in sorted order and item to knapsack if sufficient weight left.
- Bad example: Two items: $v_1 = 1$, $w_1 = 1$, $v_2 = W - 1$, $w_2 = W$. Greedy will pack item 1 and stop and get value 1 while optimum solution is to pack item 2 of value $W - 1$.

Is Greedy really bad?

Lemma

If all items have weight less than ϵW for some $\epsilon < 1$ then Greedy outputs a solution of value at least $(1 - \epsilon)OPT$.

Modified Greedy

Pick the better of the two solutions below:

- The solution of Greedy
- The heaviest value item

Modified Greedy

Pick the better of the two solutions below:

- The solution of Greedy
- The heaviest value item

Lemma

Modified Greedy outputs a solution of value at least $OPT/2$.

Modified Greedy

Pick the better of the two solutions below:

- The solution of Greedy
- The heaviest value item

Lemma

Modified Greedy outputs a solution of value at least $OPT/2$.

Can we do better?

Partial Enumeration and Greedy

Let k be some fixed integer.

```
current-best = 0
for each subset S of k items do
        if S is not feasible in knapsack
            continue
        include S in knapsack
        let W' = W − ∑_{i∈S} w_i (* remaining capacity *)
        run Greedy on remaining items in knapsack of capacity W'
        if value of solution is better than current-best
            current-best = value of new solution
end for
```

Partial Enumeration and Greedy

Let k be some fixed integer.

```
current-best = 0
for each subset S of k items do
        if S is not feasible in knapsack
            continue
        include S in knapsack
        let W' = W − ∑_{i∈S} w_i (* remaining capacity *)
        run Greedy on remaining items in knapsack of capacity W'
        if value of solution is better than current-best
            current-best = value of new solution
end for
```

Lemma

Algorithm can be implemented in $O(n^{k+1})$ time. The algorithm outputs a solution of value at least $OPT(1 - 1/k)$.

Polynomial time approximation scheme

Theorem

For the Knapsack problem, for any fixed $\epsilon > 0$, there is a $n^{O(1/\epsilon)}$-time algorithm that has an approximation ratio of $(1 - \epsilon)$.

Polynomial time approximation scheme

Theorem

For the Knapsack problem, for any fixed $\epsilon > 0$, there is a $n^{O(1/\epsilon)}$-time algorithm that has an approximation ratio of $(1 - \epsilon)$.

Knapsack has a polynomial time approximation scheme (PTAS). It is a scheme because the algorithm for each $\epsilon > 0$ is (slightly) different.

Polynomial time approximation scheme

Theorem

For the Knapsack problem, for any fixed $\epsilon > 0$, there is a $n^{O(1/\epsilon)}$-time algorithm that has an approximation ratio of $(1 - \epsilon)$.

Knapsack has a polynomial time approximation scheme (PTAS). It is a scheme because the algorithm for each $\epsilon > 0$ is (slightly) different.

Running time of algorithm for $\epsilon = 1/10$ is $O(n^{11})$; not great.

Polynomial time approximation scheme

Theorem

For the Knapsack problem, for any fixed $\epsilon > 0$, there is a $n^{O(1/\epsilon)}$-time algorithm that has an approximation ratio of $(1 - \epsilon)$.

Knapsack has a polynomial time approximation scheme (PTAS). It is a scheme because the algorithm for each $\epsilon > 0$ is (slightly) different.
Running time of algorithm for $\epsilon = 1/10$ is $O(n^{11})$; not great.

Can we do better?

Fully-polynomial Time Approximation Scheme

Theorem

For the Knapsack problem, for any fixed $\epsilon > 0$, there is an algorithm that runs in $O(n \log \frac{1}{\epsilon} + \frac{1}{\epsilon^4})$ and has an approximation ratio of $(1 - \epsilon)$.

The running time of algorithm is polynomial in both n *and* $1/\epsilon$. Such an approximation scheme is called a fully-polynomial time approximate scheme (FPTAS). This is the best we can hope for in terms of an NP-Hard optimization problem if $P \neq NP$.

Fully-polynomial Time Approximation Scheme

Theorem

For the Knapsack problem, for any fixed $\epsilon > 0$, there is an algorithm that runs in $O(n \log \frac{1}{\epsilon} + \frac{1}{\epsilon^4})$ and has an approximation ratio of $(1 - \epsilon)$.

The running time of algorithm is polynomial in both n *and* $1/\epsilon$. Such an approximation scheme is called a fully-polynomial time approximate scheme (FPTAS). This is the best we can hope for in terms of an NP-Hard optimization problem if $P \neq NP$.

Knapsack is "easy" in theory and practice even though it is NP-Complete.

Part II

Set Cover

(Weighted) Set Cover Problem

Input Given a set U of n elements, a collection $S_1, S_2, \ldots S_m$ of subsets of U, with weights w_i

Goal Find a collection $\mathcal{C}$ of these sets S_i whose union is equal to U and such that $\sum_{i \in \mathcal{C}} w_i$ is minimized.

(Weighted) Set Cover Problem

Input Given a set U of n elements, a collection $S_1, S_2, \ldots S_m$ of subsets of U, with weights w_i

Goal Find a collection $\mathcal{C}$ of these sets S_i whose union is equal to U and such that $\sum_{i \in \mathcal{C}} w_i$ is minimized.

Example

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, with

$$
\begin{array}{llll}
S_1 = \{1\} & w_1 = 1 & S_2 = \{2\} & w_2 = 1 \\
S_3 = \{3, 4\} & w_3 = 1 & S_4 = \{5, 6, 7, 8\} & w_4 = 1 \\
S_5 = \{1, 3, 5, 7\} & w_5 = 1 + \epsilon & S_6 = \{2, 4, 6, 8\} & w_6 = 1 + \epsilon
\end{array}
$$

(Weighted) Set Cover Problem

Input Given a set U of n elements, a collection $S_1, S_2, \ldots S_m$ of subsets of U, with weights w_i

Goal Find a collection $\mathcal{C}$ of these sets S_i whose union is equal to U and such that $\sum_{i \in \mathcal{C}} w_i$ is minimized.

Example

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, with

$$S_1 = \{1\} \quad w_1 = 1 \quad S_2 = \{2\} \quad w_2 = 1$$
$$S_3 = \{3, 4\} \quad w_3 = 1 \quad S_4 = \{5, 6, 7, 8\} \quad w_4 = 1$$
$$S_5 = \{1, 3, 5, 7\} \quad w_5 = 1 + \epsilon \quad S_6 = \{2, 4, 6, 8\} \quad w_6 = 1 + \epsilon$$

$\{S_5, S_6\}$ is a set cover of weight $2 + 2\epsilon$

Greedy Rule

- Pick the next set in the cover to be the one that makes "most progress" towards the goal

Greedy Rule

- Pick the next set in the cover to be the one that makes "most progress" towards the goal
  - Covers many (uncovered) elements

Greedy Rule

- Pick the next set in the cover to be the one that makes "most progress" towards the goal
  - Covers many (uncovered) elements
  - Has a small weight

Greedy Rule

- Pick the next set in the cover to be the one that makes "most progress" towards the goal
  - Covers many (uncovered) elements
  - Has a small weight
- If R is the set of elements that aren't covered as yet, add set S_i to the cover, if it minimizes the quantity $\frac{w_i}{|S_i \cap R|}$; that is the set that maximizes the ratio of weight to number of new elements covered.

Greedy Rule

- Pick the next set in the cover to be the one that makes "most progress" towards the goal
  - Covers many (uncovered) elements
  - Has a small weight
- If R is the set of elements that aren't covered as yet, add set S_i to the cover, if it minimizes the quantity $\frac{w_i}{|S_i \cap R|}$; that is the set that maximizes the ratio of weight to number of new elements covered.
- If all $w_i = 1$ then greedy picks the next set that covers the max number of uncovered elements.

Greedy Algorithm

```
Initially R = U and C = ∅
while R ≠ ∅
    let S_i be the set that minimizes w_i/|S_i ∩ R|
    C = C ∪ {i}
    R = R \ S_i
return C
```

Running Time

Greedy Algorithm

```
Initially R = U and C = ∅
while R ≠ ∅
    let S_i be the set that minimizes w_i/|S_i ∩ R|
    C = C ∪ {i}
    R = R \ S_i
return C
```

Running Time

Greedy Algorithm

```
Initially R = U and C = ∅
while R ≠ ∅
    let S_i be the set that minimizes w_i/|S_i ∩ R|
    C = C ∪ {i}
    R = R \ S_i
return C
```

Running Time

- Main loop iterates for $O(n)$ time, where $|U| = n$

Greedy Algorithm

```
Initially R = U and C = ∅
while R ≠ ∅
    let S_i be the set that minimizes w_i/|S_i ∩ R|
    C = C ∪ {i}
    R = R \ S_i
return C
```

Running Time

- Main loop iterates for $O(n)$ time, where $|U| = n$

Greedy Algorithm

```
Initially R = U and C = ∅
while R ≠ ∅
    let S_i be the set that minimizes w_i/|S_i ∩ R|
    C = C ∪ {i}
    R = R \ S_i
return C
```

Running Time

- Main loop iterates for $O(n)$ time, where $|U| = n$
- Minimum S_i can be found in $O(\log m)$ time, using a priority heap, where there are m sets in set cover instance

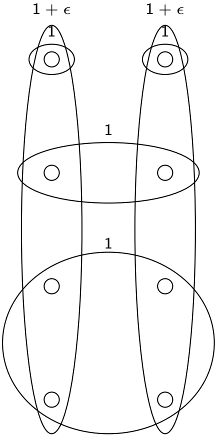
Greedy Algorithm

```
Initially R = U and C = ∅
while R ≠ ∅
    let S_i be the set that minimizes w_i/|S_i ∩ R|
    C = C ∪ {i}
    R = R \ S_i
return C
```

Running Time

- Main loop iterates for $O(n)$ time, where $|U| = n$
- Minimum S_i can be found in $O(\log m)$ time, using a priority heap, where there are m sets in set cover instance
- Total time is $O(n \log m)$

Example: Greedy Algorithm



Example

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, with

$$\begin{array}{ll} S_1 = \{1\} & S_2 = \{2\} \\ S_3 = \{3, 4\} & S_4 = \{5, 6, 7, 8\} \\ S_5 = \{1, 3, 5, 7\} & S_6 = \{2, 4, 6, 8\} \end{array}$$

$w_1 = w_2 = w_3 = w_4 = 1$ and $w_5 = w_6 = 1 + \epsilon$
Greedy Algorithm first picks S_4, then S_3, and finally S_1 and S_2

Analysis of the Greedy Algorithm

$H(k)$: k'th harmonic number. $H(k) = 1 + 1/2 + \ldots + 1/k \simeq \ln k$.

Theorem

The greedy algorithm for set cover is a $H(d^*)$-approximation, where $d^* = \max_i |S_i|$

Analysis of the Greedy Algorithm

$H(k)$: k'th harmonic number. $H(k) = 1 + 1/2 + \ldots + 1/k \simeq \ln k$.

Theorem

The greedy algorithm for set cover is a $H(d^*)$-approximation, where $d^* = \max_i |S_i|$

Analysis Tight?

Does the Greedy Algorithm give better approximation guarantees?

Analysis of the Greedy Algorithm

$H(k)$: k'th harmonic number. $H(k) = 1 + 1/2 + \ldots + 1/k \simeq \ln k$.

Theorem

The greedy algorithm for set cover is a $H(d^*)$-approximation, where $d^* = \max_i |S_i|$

Analysis Tight?

Does the Greedy Algorithm give better approximation guarantees? No!

Consider a generalization of the set cover example. Each column has 2^{k-1} elements, and there are two sets consisting of a column each with weight $1 + \epsilon$. Additionally there are $\log n$ sets of increasing size of weight 1. The greedy algorithm will pick these $\log n$ sets given weight $\log n$, while the best cover has weight $2 + 2\epsilon$

Best Algorithm for Set Cover

Theorem

If $P \neq NP$ then no polynomial time algorithm can achieve a better than $H(n)$ approximation.

Proof beyond the scope of this course.

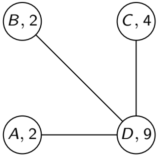
Part III

Vertex Cover

(Weighted) Vertex Cover

Input Given graph $G = (V, E)$ with weights $w_i \geq 0$ associated with each vertex i

Goal Find a vertex cover $S \subseteq V$ such that $\sum_{i \in S} w_i$ is minimized



(Weighted) Vertex Cover

Input Given graph $G = (V, E)$ with weights $w_i \geq 0$ associated with each vertex i

Goal Find a vertex cover $S \subseteq V$ such that $\sum_{i \in S} w_i$ is minimized

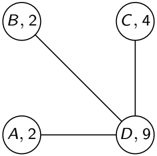


Figure: Vertex Cover $\{D\}$ has weight 9, while $\{A, B, C\}$ has weight 8

Unweighted Case

Question

When all sets have weight 1 and all vertices have weight 1, we know Vertex Cover $\leq_P$ Set Cover
Can we use the approximation algorithm for Set Cover to get an approximation algorithm for Vertex Cover?

Unweighted Case

Question

When all sets have weight 1 and all vertices have weight 1, we know Vertex Cover $\leq_P$ Set Cover
Can we use the approximation algorithm for Set Cover to get an approximation algorithm for Vertex Cover? Yes, but not true for all reductions

Approximating Vertex Cover using Set Cover

Theorem

There is an $H(d)$-approximation algorithm for Vertex Cover, where d is the maximum degree of any vertex

Approximating Vertex Cover using Set Cover

Theorem

There is an $H(d)$-approximation algorithm for Vertex Cover, where d is the maximum degree of any vertex

Proof.

The approximation algorithm uses the reduction to set cover

Approximating Vertex Cover using Set Cover

Theorem

There is an $H(d)$-approximation algorithm for Vertex Cover, where d is the maximum degree of any vertex

Proof.

The approximation algorithm uses the reduction to set cover

- The universe (for set cover problem) is the set of edges

Approximating Vertex Cover using Set Cover

Theorem

There is an $H(d)$-approximation algorithm for Vertex Cover, where d is the maximum degree of any vertex

Proof.

The approximation algorithm uses the reduction to set cover

- The universe (for set cover problem) is the set of edges
- For each vertex i, create S_i of edges incident upon i, with weight w_i

Approximating Vertex Cover using Set Cover

Theorem

There is an $H(d)$-approximation algorithm for Vertex Cover, where d is the maximum degree of any vertex

Proof.

The approximation algorithm uses the reduction to set cover

- The universe (for set cover problem) is the set of edges
- For each vertex i, create S_i of edges incident upon i, with weight w_i
- $\mathcal{C}$ is a set cover of weight w iff $\mathcal{C}$ is a vertex cover of weight w

Approximating Vertex Cover using Set Cover

Theorem

There is an $H(d)$-approximation algorithm for Vertex Cover, where d is the maximum degree of any vertex

Proof.

The approximation algorithm uses the reduction to set cover

- The universe (for set cover problem) is the set of edges
- For each vertex i, create S_i of edges incident upon i, with weight w_i
- $\mathcal{C}$ is a set cover of weight w iff $\mathcal{C}$ is a vertex cover of weight w
- The desired approximation algorithm runs the algorithm for set cover on the reduced instance □

Greedy for Vertex Cover

Reduction essentially says to run following greedy algorithm:

- Initialize S to $\emptyset$.
- While graph has edges left
  - Pick vertex v with highest degree and add it to S
  - Remove v and all edges incident to v from graph
- Ouput S

Reductions and Approximation Algorithms

Polynomial reduction does not imply approximation

Reductions and Approximation Algorithms

Polynomial reduction does not imply approximation

- Independent Set $\leq_P$ Vertex Cover

Reductions and Approximation Algorithms

Polynomial reduction does not imply approximation

- Independent Set $\leq_P$ Vertex Cover
  - $V \setminus I$ is a vertex cover iff I is independent set

Reductions and Approximation Algorithms

Polynomial reduction does not imply approximation

- Independent Set $\leq_P$ Vertex Cover
  - $V \setminus I$ is a vertex cover iff I is independent set
- Suppose we have a 2-approximation alg $\mathcal{A}$ for vertex cover

Reductions and Approximation Algorithms

Polynomial reduction does not imply approximation

- Independent Set $\leq_P$ Vertex Cover
  - $V \setminus I$ is a vertex cover iff I is independent set
- Suppose we have a 2-approximation alg $\mathcal{A}$ for vertex cover
- Suppose $G = (V, E)$ has vertex cover of size $|V|/2$

Reductions and Approximation Algorithms

Polynomial reduction does not imply approximation

- Independent Set $\leq_P$ Vertex Cover
  - $V \setminus I$ is a vertex cover iff I is independent set
- Suppose we have a 2-approximation alg $\mathcal{A}$ for vertex cover
- Suppose $G = (V, E)$ has vertex cover of size $|V|/2$
- So $\mathcal{A}$ could return V as vertex cover on G

Reductions and Approximation Algorithms

Polynomial reduction does not imply approximation

- Independent Set $\leq_P$ Vertex Cover
  - $V \setminus I$ is a vertex cover iff I is independent set
- Suppose we have a 2-approximation alg $\mathcal{A}$ for vertex cover
- Suppose $G = (V, E)$ has vertex cover of size $|V|/2$
- So $\mathcal{A}$ could return V as vertex cover on G
- $V \setminus V = \emptyset$ is not a good approximation of an independent set of size $|V|/2$

Back to Vertex Cover

Greedy algorithm given an $H(n) \simeq \ln n$ approximation.

Back to Vertex Cover

Greedy algorithm given an $H(n) \simeq \ln n$ approximation.

- Does Greedy actually give a better approximation?

Back to Vertex Cover

Greedy algorithm given an $H(n) \simeq \ln n$ approximation.

- Does Greedy actually give a better approximation? No!. There are examples which show that Greedy is no better than $H(n)$ approx.

Back to Vertex Cover

Greedy algorithm given an $H(n) \simeq \ln n$ approximation.

- Does Greedy actually give a better approximation? No!. There are examples which show that Greedy is no better than $H(n)$ approx.
- Is there a better approximation algorithm for Vertex Cover?

Back to Vertex Cover

Greedy algorithm given an $H(n) \simeq \ln n$ approximation.

- Does Greedy actually give a better approximation? No!. There are examples which show that Greedy is no better than $H(n)$ approx.
- Is there a better approximation algorithm for Vertex Cover? Yes! We will see a 2-approximation.

Vertex Cover as Linear Constraints

For a graph $G = (V, E)$ with vertex weights w_i, we have variables x_i, which will be either 0 or 1, indicating whether vertex i is part of the cover

$$\begin{array}{lll} \text{Minimize} & \sum_{i \in V} w_i x_i & \\ \text{subject to} & x_i + x_j \geq 1 & \text{for each } (i,j) \in E \\ & x_i \in \{0,1\} & \text{for each } i \in V \end{array}$$

0-1 Integer Linear Programming

Integer Programming

Given an objective function, and a collection of linear constraints, find an assignment of 0-1 values to the variables such that all linear constraints are satisfied and the objective function is optimized.

0-1 Integer Linear Programming

Integer Programming

Given an objective function, and a collection of linear constraints, find an assignment of 0-1 values to the variables such that all linear constraints are satisfied and the objective function is optimized.

Theorem

0-1 Integer Linear programming is NP-complete

LP Relaxation

The linear programming relaxation of an Integer Linear Program is obtained by removing the constraint that the variables be integers

$$
\begin{array}{lll}
\text{Minimize} & \sum_{i \in V} w_i x_i & \\
\text{subject to} & x_i + x_j \geq 1 & \text{for each } (i,j) \in E \\
 & x_i \geq 0 & \text{for each } i \in V
\end{array}
$$

LP Relaxation

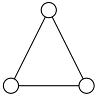
The linear programming relaxation of an Integer Linear Program is obtained by removing the constraint that the variables be integers

$$\begin{array}{lll} \text{Minimize} & \sum_{i \in V} w_i x_i & \\ \text{subject to} & x_i + x_j \geq 1 & \text{for each } (i,j) \in E \\ & x_i \geq 0 & \text{for each } i \in V \end{array}$$

Proposition

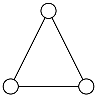
If x^1 is the optimal solution for the LP relaxation, and x^2 is the optimal solution to the ILP, then $\sum_{i \in V} w_i x_i^1 \leq \sum_{i \in V} w_i x_i^2$

LP Relaxation: Example



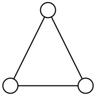
All vertices have weight 1.

LP Relaxation: Example



All vertices have weight 1. Any vertex cover must have at least 2 vertices, and hence weight 2

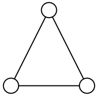
LP Relaxation: Example



$$
\begin{aligned}
\text{Min} \quad & x_1 + x_2 + x_3 \\
\text{s.t.} \quad & x_1 + x_2 \geq 1 \\
& x_2 + x_3 \geq 1 \\
& x_3 + x_1 \geq 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

All vertices have weight 1. Any vertex cover must have at least 2 vertices, and hence weight 2

LP Relaxation: Example



$$\begin{aligned} \text{Min} \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_3 + x_1 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

All vertices have weight 1. Any vertex cover must have at least 2 vertices, and hence weight 2

The LP problem has solution $x_1 = x_2 = x_3 = 1/2$ whose value is $3/2$

Vertex Cover: LP Relaxation

Weighted vertex cover is the ILP

$$\begin{array}{lll} \text{Min} & \sum_{i \in V} w_i x_i & \\ \text{s.t.} & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \in \{0,1\} & i \in V \end{array}$$

Vertex Cover: LP Relaxation

Weighted vertex cover is the ILP

$$\begin{array}{lll} \text{Min} & \sum_{i \in V} w_i x_i & \\ \text{s.t.} & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \in \{0,1\} & i \in V \end{array}$$

Its LP-relaxation is

$$\begin{array}{lll} \text{Min} & \sum_{i \in V} w_i x_i & \\ \text{s.t} & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \geq 0 & i \in V \end{array}$$

Vertex Cover: LP Relaxation

Weighted vertex cover is the ILP

$$\begin{array}{lll} \text{Min} & \sum_{i \in V} w_i x_i & \\ \text{s.t.} & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \in \{0,1\} & i \in V \end{array}$$

Its LP-relaxation is

$$\begin{array}{lll} \text{Min} & \sum_{i \in V} w_i x_i & \\ \text{s.t} & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \geq 0 & i \in V \end{array}$$

- Solutions to the LP don't correspond to vertex covers, because variables may have fractional values.

Vertex Cover: LP Relaxation

Weighted vertex cover is the ILP

$$\begin{array}{lll} \text{Min} & \sum_{i \in V} w_i x_i & \\ \text{s.t.} & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \in \{0,1\} & i \in V \end{array}$$

Its LP-relaxation is

$$\begin{array}{lll} \text{Min} & \sum_{i \in V} w_i x_i & \\ \text{s.t} & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \geq 0 & i \in V \end{array}$$

- Solutions to the LP don't correspond to vertex covers, because variables may have fractional values.
- Can solving the LP-relaxation, nonetheless, help?

LP rounding

Algorithm

LP rounding

Algorithm

1. Solve LP relaxation optimally to get solution x^*

LP rounding

Algorithm

1. Solve LP relaxation optimally to get solution x^*
2. Round the fraction values to obtain a solution to the vertex cover problem, i.e., $S = \{i \mid x_i^* \geq 1/2\}$

LP rounding

Algorithm

1. Solve LP relaxation optimally to get solution x^*
2. Round the fraction values to obtain a solution to the vertex cover problem, i.e., $S = \{i \mid x_i^* \geq 1/2\}$
3. return S

LP rounding

Algorithm

1. Solve LP relaxation optimally to get solution x^*
2. Round the fraction values to obtain a solution to the vertex cover problem, i.e., $S = \{i \mid x_i^* \geq 1/2\}$
3. return S

Challenges

LP rounding

Algorithm

1. Solve LP relaxation optimally to get solution x^*
2. Round the fraction values to obtain a solution to the vertex cover problem, i.e., $S = \{i \mid x_i^* \geq 1/2\}$
3. return S

Challenges

- Is S obtained by rounding, guaranteed to be a vertex cover?

LP rounding

Algorithm

1. Solve LP relaxation optimally to get solution x^*
2. Round the fraction values to obtain a solution to the vertex cover problem, i.e., $S = \{i \mid x_i^* \geq 1/2\}$
3. return S

Challenges

- Is S obtained by rounding, guaranteed to be a vertex cover?
- How large is $w(S)$ compared to optimal cover?

Correctness of LP rounding

Lemma

Set S obtained by rounding the LP-relaxation is a vertex cover

Correctness of LP rounding

Lemma

Set S obtained by rounding the LP-relaxation is a vertex cover

Proof.

Correctness of LP rounding

Lemma

Set S obtained by rounding the LP-relaxation is a vertex cover

Proof.

- Consider any edge $e = (i, j)$

Correctness of LP rounding

Lemma

Set S obtained by rounding the LP-relaxation is a vertex cover

Proof.

- Consider any edge $e = (i, j)$
- Since $x_i^* + x_j^* \geq 1$, we know $x_i^* \geq 1/2$ or $x_j^* \geq 1/2$

Correctness of LP rounding

Lemma

Set S obtained by rounding the LP-relaxation is a vertex cover

Proof.

- Consider any edge $e = (i, j)$
- Since $x_i^* + x_j^* \geq 1$, we know $x_i^* \geq 1/2$ or $x_j^* \geq 1/2$
- Thus, either i or j is in S □

Lower Bound provided by LP Relaxation

Lemma

Let w^*_{LP} be the optimal value of the LP relaxation and let w^* be the weight of an optimum vertex cover. Then $w^*_{LP} \leq w^*$.

Lower Bound provided by LP Relaxation

Lemma

Let w^*_{LP} be the optimal value of the LP relaxation and let w^* be the weight of an optimum vertex cover. Then $w^*_{LP} \leq w^*$.

Proof.

Let S^* be an optimum vertex cover with weight w^*.
Consider a *feasible solution* x to LP where $x_i = 1$ if $i \in S$ and $x_i = 0$ otherwise. $wx = w(S^*) = w^*$.
Therefore optimum value of LP can be no more than w^*. □

Lower Bound provided by LP Relaxation

Lemma

Let w_{LP}^* be the optimal value of the LP relaxation and let w^* be the weight of an optimum vertex cover. Then $w_{LP}^* \leq w^*$.

Proof.

Let S^* be an optimum vertex cover with weight w^*.
Consider a *feasible solution* x to LP where $x_i = 1$ if $i \in S$ and $x_i = 0$ otherwise. $wx = w(S^*) = w^*$.
Therefore optimum value of LP can be no more than w^*. □

Thus solving the LP gives us a lower bound on the weight on w^*. This is often useful in also back-tracking heuristics that solve the problem exactly in exponential time.

Approximation Guarantee

Lemma

Weight of vertex cover S return by LP rounding is at most 2-times the weight of the optimal cover

Approximation Guarantee

Lemma

Weight of vertex cover S return by LP rounding is at most 2-times the weight of the optimal cover

Proof.

Let w^*_{LP} be the optimal value for the LP, and let S be cover obtained by rounding x^*. Let S^* be the optimum vertex cover

Approximation Guarantee

Lemma

Weight of vertex cover S return by LP rounding is at most 2-times the weight of the optimal cover

Proof.

Let w^*_{LP} be the optimal value for the LP, and let S be cover obtained by rounding x^*. Let S^* be the optimum vertex cover

- $w^*_{LP} \leq w(S^*)$ from lemma.

Approximation Guarantee

Lemma

Weight of vertex cover S return by LP rounding is at most 2-times the weight of the optimal cover

Proof.

Let w^*_{LP} be the optimal value for the LP, and let S be cover obtained by rounding x^*. Let S^* be the optimum vertex cover

- $w^*_{LP} \leq w(S^*)$ from lemma.
- $w^*_{LP} = \sum_i w_i x^*_i \geq \sum_{i \in S} w_i x^*_i$

Approximation Guarantee

Lemma

Weight of vertex cover S return by LP rounding is at most 2-times the weight of the optimal cover

Proof.

Let w^*_{LP} be the optimal value for the LP, and let S be cover obtained by rounding x^*. Let S^* be the optimum vertex cover

- $w^*_{LP} \leq w(S^*)$ from lemma.
- $w^*_{LP} = \sum_i w_i x^*_i \geq \sum_{i \in S} w_i x^*_i$
- Since $x^*_i \geq 1/2$ for every $i \in S$, we have $w^*_{LP} \geq \sum_{i \in S} w_i x^*_i \geq (1/2) \sum_{i \in S} w_i = (1/2) w(S)$

Approximation Guarantee

Lemma

Weight of vertex cover S return by LP rounding is at most 2-times the weight of the optimal cover

Proof.

Let w^*_{LP} be the optimal value for the LP, and let S be cover obtained by rounding x^*. Let S^* be the optimum vertex cover

- $w^*_{LP} \leq w(S^*)$ from lemma.
- $w^*_{LP} = \sum_i w_i x^*_i \geq \sum_{i \in S} w_i x^*_i$
- Since $x^*_i \geq 1/2$ for every $i \in S$, we have $w^*_{LP} \geq \sum_{i \in S} w_i x^*_i \geq (1/2) \sum_{i \in S} w_i = (1/2) w(S)$
- Hence $w(S) \leq 2w^*_{LP} \leq 2w(S^*)$. □

Integer Programming and Heuristics

Using Integer Programming to solve problems is a a *meta-heuristic* method.

- integer programming can "naturally" model many NP-Complete problems
- linear programming relaxations and a variety of heuristic methods like branch-and-bound, branch-and-cut, cutting places, etc are used to solve integer programs in practice
- very effective for many applications

Approximation and NP-Hard problems

- Load balancing: can obtain a $(1+\epsilon)$-approximation in $n^{O(1/\epsilon)}$) time. PTAS
- Knapsack: can obtain a $(1-\epsilon)$-approximation in $O(n \log 1/\epsilon + 1/\epsilon^4)$ time. FPTAS
- Set cover: can obtain an $\ln n + 1$ approximation. Essentially no better approximation possible unless $P = NP$
- Vertex cover: 2-approximation. Unless $P = NP$ cannot obtain a 1.36 approximation.

Approximation and NP-Hard problems

- Load balancing: can obtain a $(1+\epsilon)$-approximation in $n^{O(1/\epsilon)}$) time. PTAS
- Knapsack: can obtain a $(1-\epsilon)$-approximation in $O(n \log 1/\epsilon + 1/\epsilon^4)$ time. FPTAS
- Set cover: can obtain an $\ln n + 1$ approximation. Essentially no better approximation possible unless $P = NP$
- Vertex cover: 2-approximation. Unless $P = NP$ cannot obtain a 1.36 approximation.

Lesson: NP-Hard optimization problems can differ dramatically in approximation even though they are all equivalent in terms of exact solvability. Some are (much) easier than others.

Approximation Algorithms: Pros and Cons

Pros:

- Systematic and theoretically sound approach to studying heuristics for problems
- Explanation for why/how problems differ despite equivalence for exact solvability
- Allows one to explore structure of intractable problems
- Can lead to successful heuristics
- Algorithmic and mathematical elegance

Cons:

- Not applicable to decision problems such as SAT
- Worst-case approach is not ideal in some practical situations