CS 473: Algorithms

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Part I

Introduction to Reductions

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Also, the right reductions might win you a million dollars!

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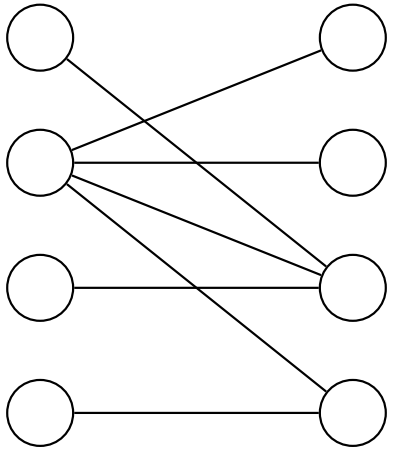

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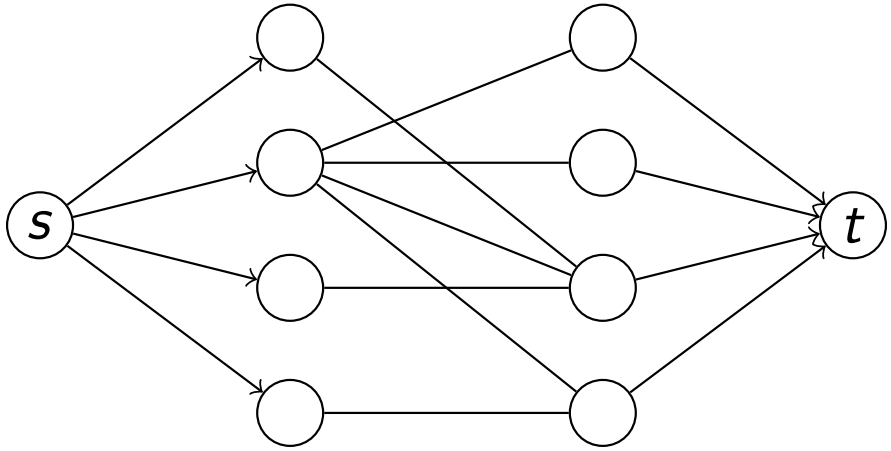

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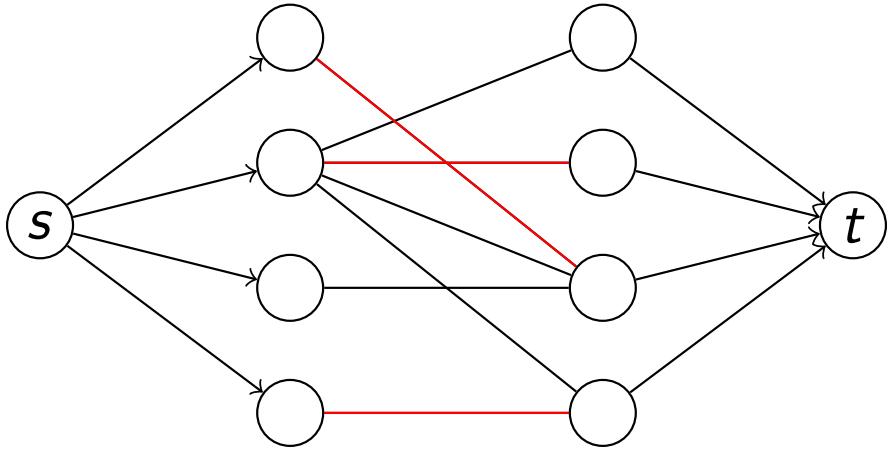

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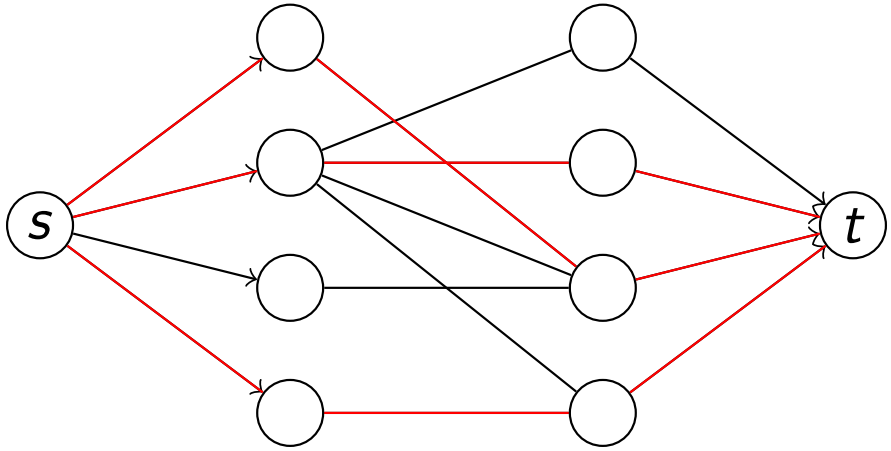

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While using reductions and comparing problems, we typically work with the decision versions. Decision problems have Yes/No answers. This makes them easy to work with.

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What is an Algorithm for a decision Problem X? It takes as input an instance of X, and outputs either "YES" or "NO".

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- Given L, Π_L is the following problem: given $x \in \Sigma^*$, is $x \in L$? Each string in Σ^* is an instance of Π_L and L is the set of YES instances.
- Given Π the associated language $L_\Pi = \{I \mid I \text{ is a YES instance of } \Pi\}$.

Thus, decision problems and languages are used interchangeably.

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(Actually, this is only one type of reduction, but this is the one we'll use most often.)

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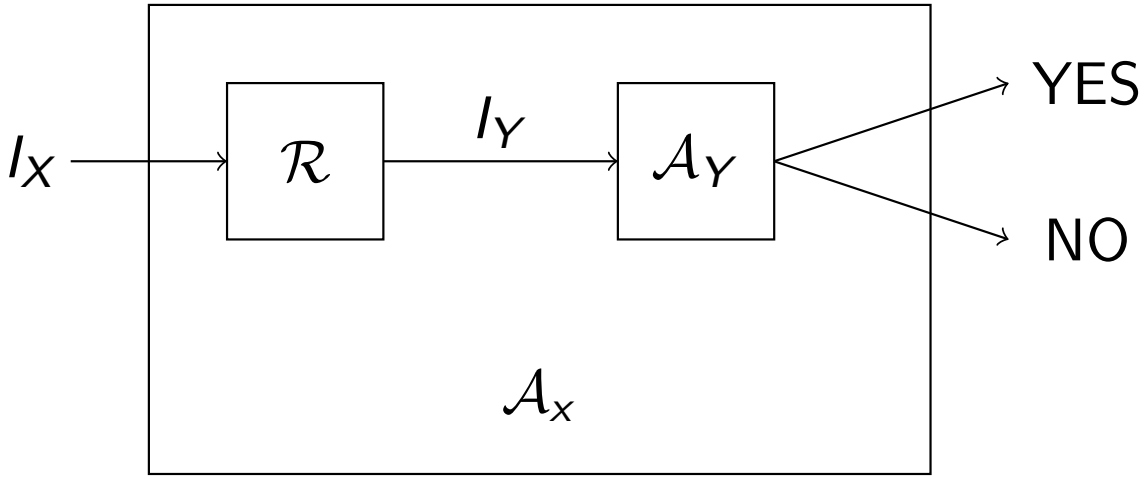

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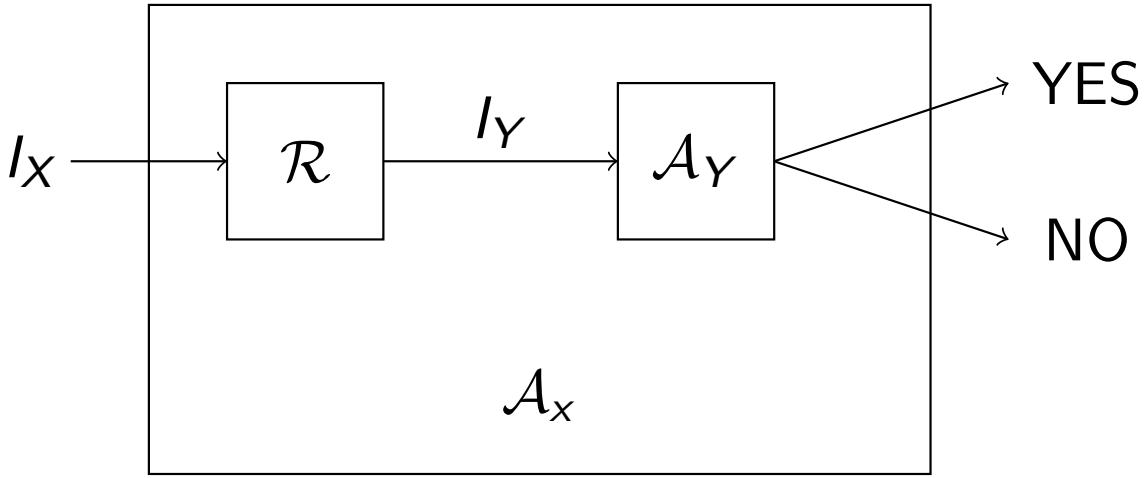

In particular, if $\mathcal{R}$ and $\mathcal{A}_Y$ are polynomial-time algorithms, $\mathcal{A}_X$ is also polynomial-time.

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- Bipartite Matching $\leq$ Max-Flow. Therefore, Bipartite Matching cannot be harder than Max-Flow.
- Equivalently, Max-Flow is at least as hard as Bipartite Matching.
- More generally, if $X \leq Y$, we can say that X is no harder than Y, or Y is at least as hard as X.

Part II

Examples of Reductions

Independent Sets and Cliques

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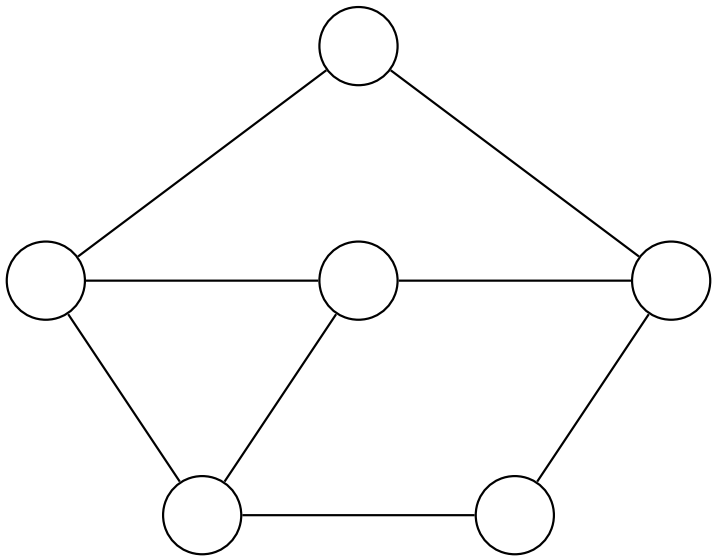

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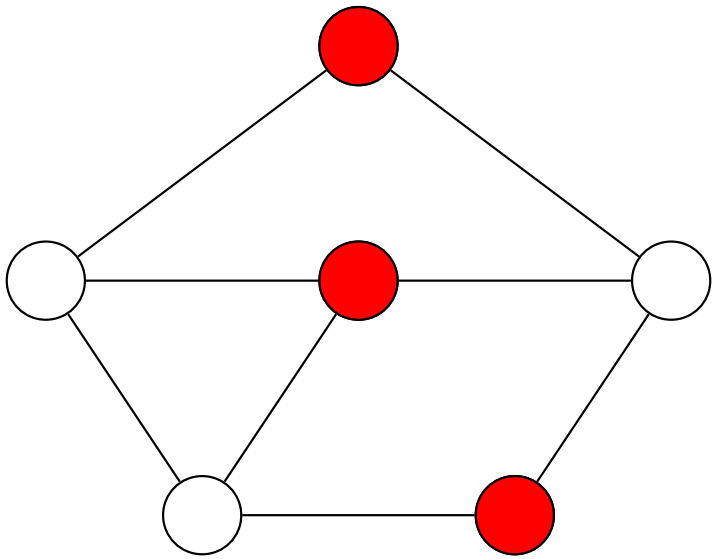

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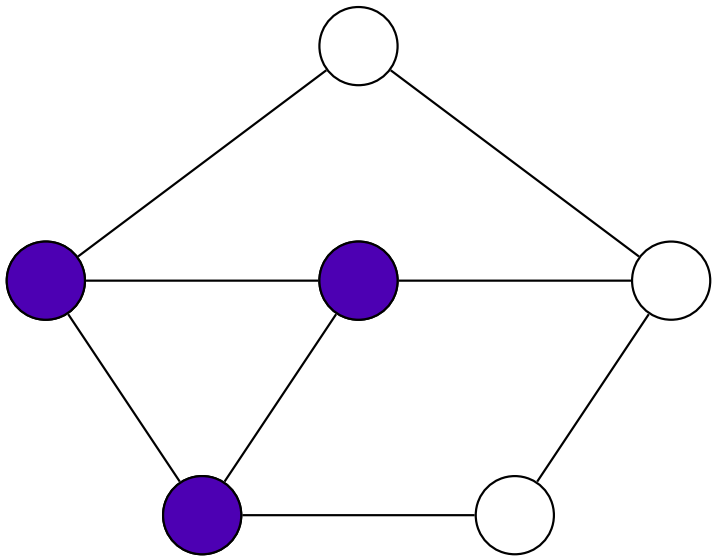

The Independent Set and Clique Problems

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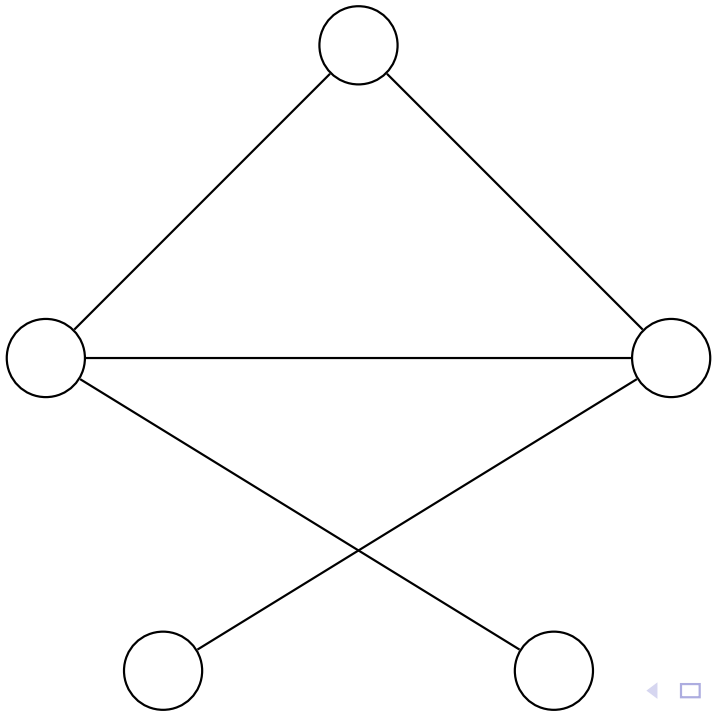

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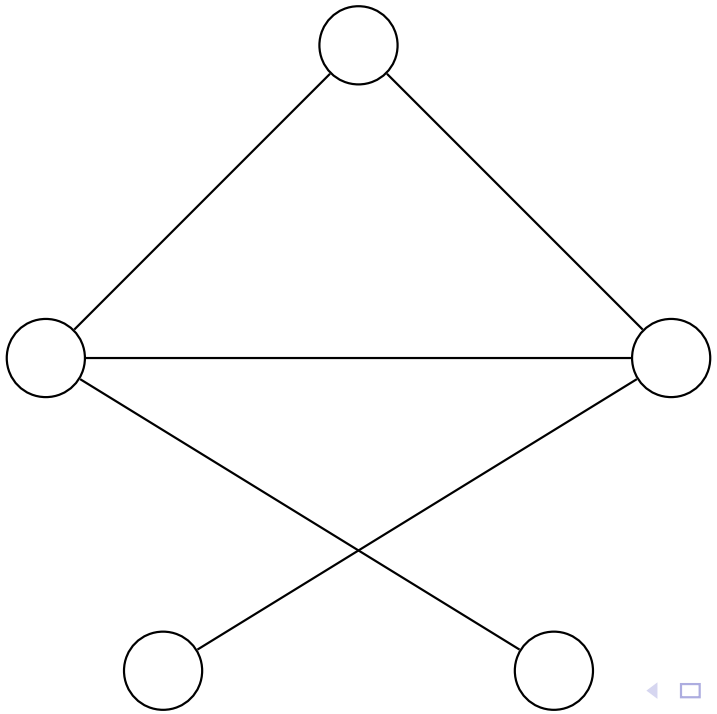

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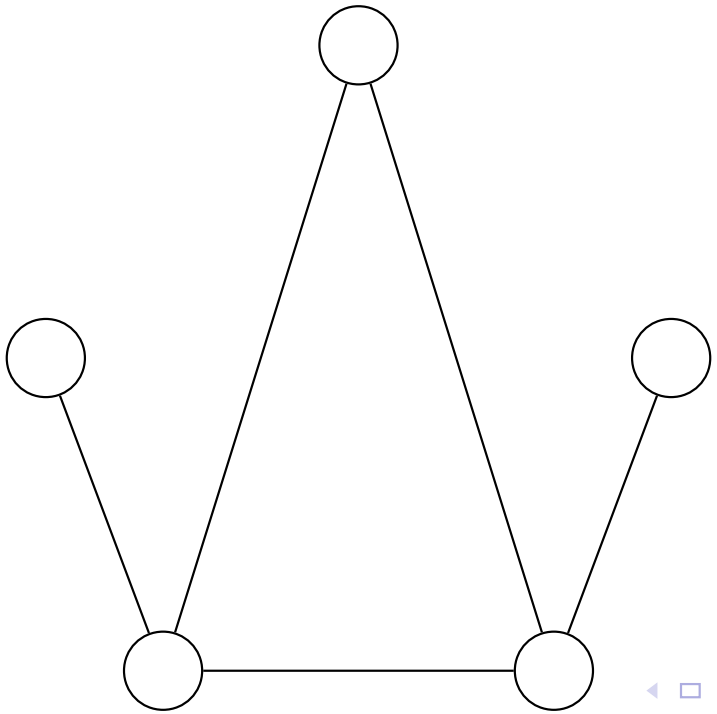

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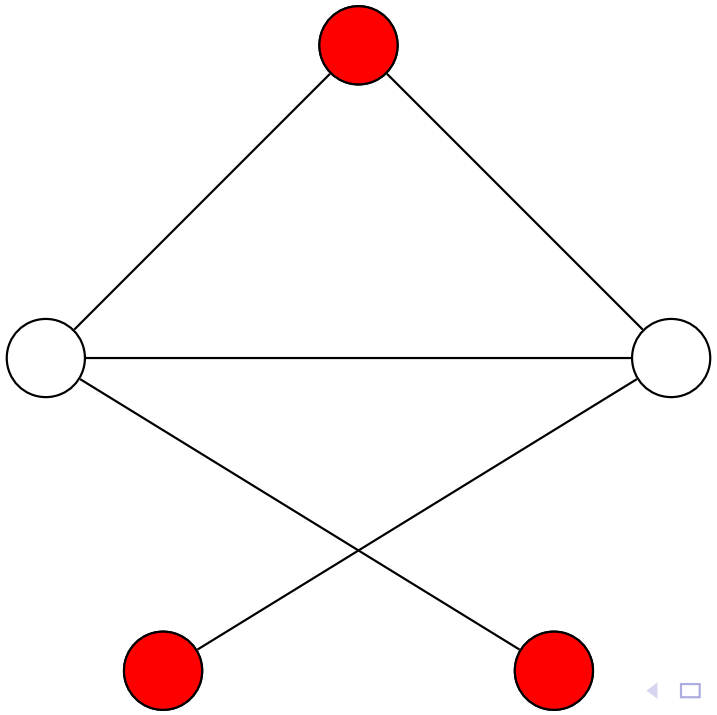

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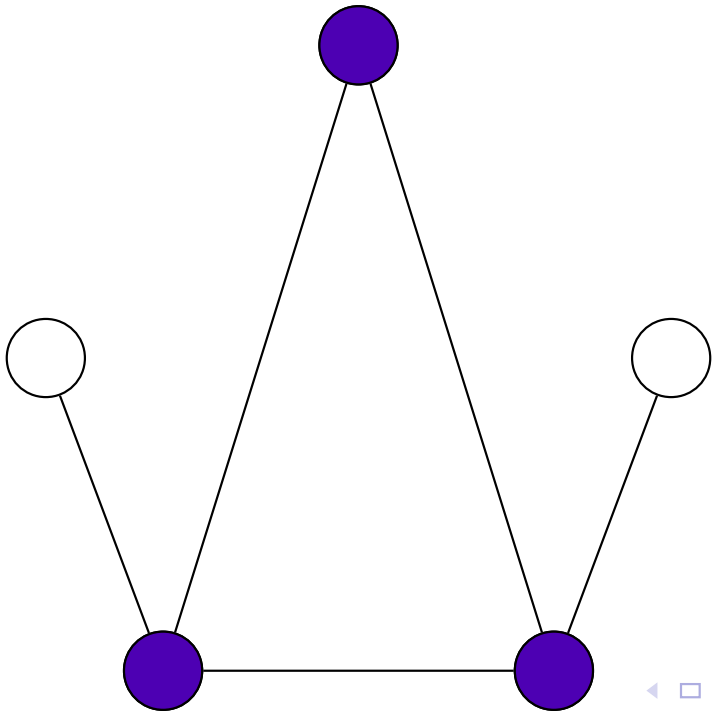

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The Clique Problem is *at least as hard as* the Independent Set problem.

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The smallest DFA equivalent to an NFA with n states may have $\approx 2^n$ states.

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We check if M has *any* reachable non-final state.
Alternatively, minimize M to obtain M' and see if M' has a single state which is an accepting state.

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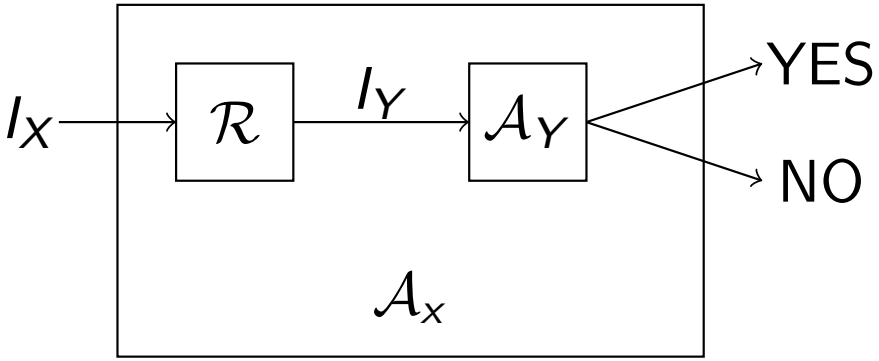

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If $X \leq_P Y$ and X does not have an efficient algorithm, Y cannot have an efficient algorithm!

Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.
I_Y is the output of $\mathcal{R}$ on input I_X
$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. $\square$

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Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

Transitivity of Reductions

Proposition

$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.

Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y In other words show that an algorithm for Y implies an algorithm for X.

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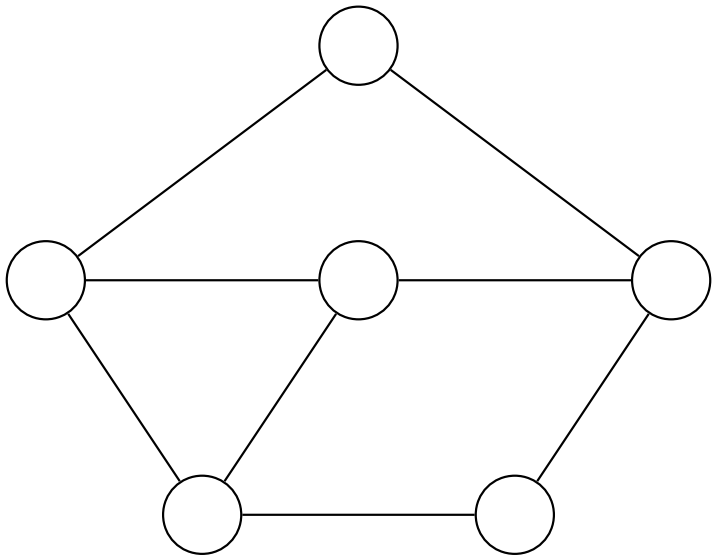

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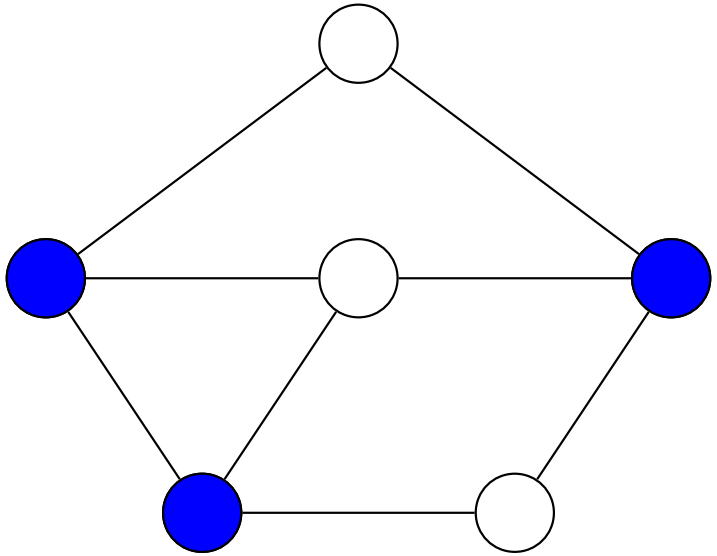

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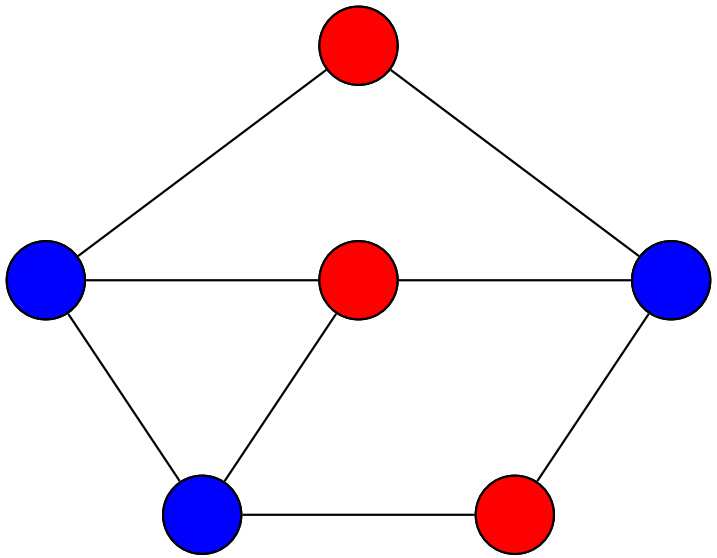

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Can we relate Independent Set and Vertex Cover?

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- (u, v) is not edge, as otherwise $V \setminus S$ does not cover (u, v)
- S is thus an independent set $\square$

Independent Set $\leq_P$ Vertex Cover

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Therefore, Independent Set $\leq_P$ Vertex Cover. Also Vertex Cover $\leq_P$ Independent Set.

A problem of Languages

Suppose you work for the United Nations. Let U be the set of all languages spoken by people across the world. The United Nations also has a set of translators, all of whom speak English, and some other languages from U.

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More General problem: Find/Hire a small group of people who can accomplish a large number of tasks.

The Set Cover Problem

Input Given a set U of n elements, a collection $S_1, S_2, \ldots S_m$ of subsets of U, and an integer k

Goal Is there is a collection of at most k of these sets S_i whose union is equal to U?

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Example

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$ with

$$\begin{array}{ll} S_1 = \{3, 7\} & S_2 = \{3, 4, 5\} \\ S_3 = \{1\} & S_4 = \{2, 4\} \\ S_5 = \{5\} & S_6 = \{1, 2, 6, 7\} \end{array}$$

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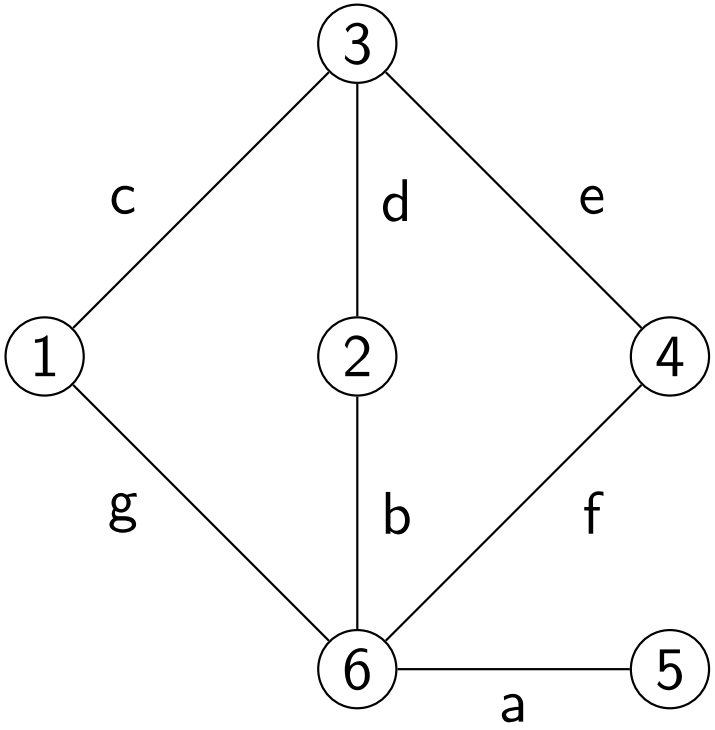
Observe that G has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

Vertex Cover $\leq_P$ Set Cover: Example



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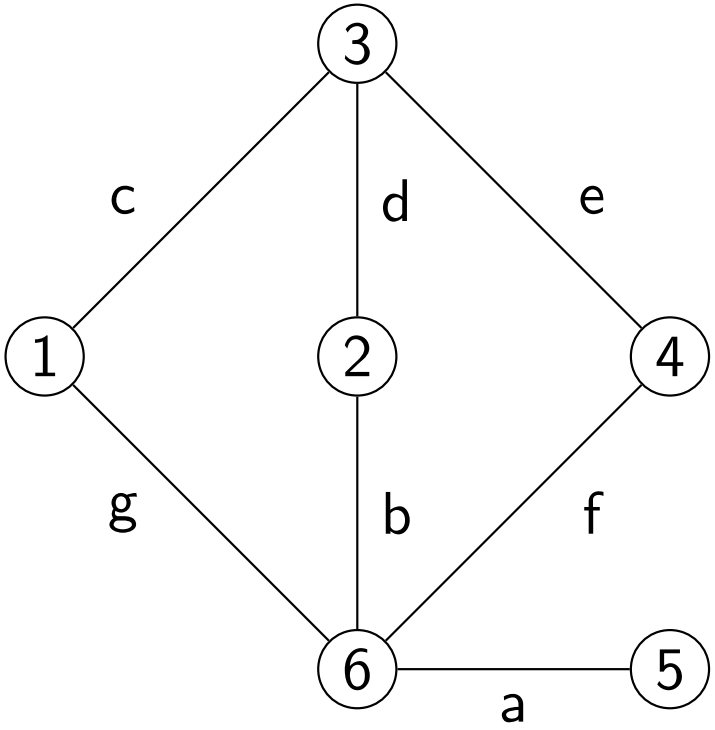

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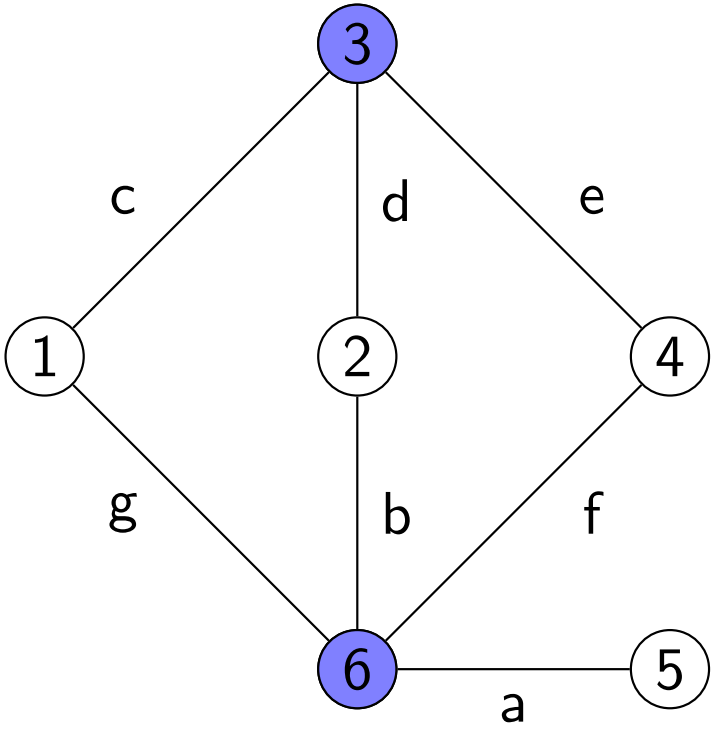

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Integer Linear Programming

Problem

Find a vector $x \in Z^d$ (integer values) that

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for } i = 1 \ldots n \end{array}$$

Input is matrix $A = (a_{ij}) \in Q^{n \times d}$, column vector $b = (b_i) \in Q^n$, and row vector $c = (c_j) \in Q^d$ where Q is the set of rational numbers.

Decision version: Given A, b, c and $v \in Q$, is the optimum of the integer program at least v?

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$$\max \sum_{v \in V} x_v$$

$$\begin{array}{rcll} x_u + x_v & \leq & 1 & \text{for each } (u,v) \in E \\ 0 \leq x_v & \leq & 1 & \text{for each } v \in V \\ x_v & & \text{is an integer for each } v \in V & \end{array}$$

Opt value of ILP instance is k iff G has indep set of size k

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\begin{aligned}
\min \sum_{v \in V} x_i & & & & \\
\sum_{i: j \in S_i} x_i & \geq & 1 & & 1 \leq j \leq n \\
0 \leq x_i & \leq & 1 & & 1 \leq i \leq m \\
x_i & & \text{is an integer for} & & 1 \leq i \leq m
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Opt value of ILP instance is k iff there is a set cover of size k

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We looked at some examples of reductions between Independent Set, Clique, Vertex Cover, Set Cover, ILP etc.