Part I

Introduction to Reductions
A reduction from Problem X to Problem Y means (informally) that if we have an algorithm for Problem Y, we can use it to find an algorithm for Problem X.
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Using Reductions

- We use reductions to find algorithms to solve problems.
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- We also use reductions to show that we can’t find algorithms for some problems. (We say that these problems are hard.)
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**Using Reductions**

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- We also use reductions to show that we can’t find algorithms for some problems. (We say that these problems are hard.)

Also, the right reductions might win you a million dollars!
Example 1: Bipartite Matching and Flows

How do we solve the Bipartite Matching Problem?

Given a bipartite graph $G = (U \cup V, E)$ and number $k$, does $G$ have a matching of size $\geq k$?

Solution

Reduce it to Max-Flow. $G$ has a matching of size $\geq k$ iff there is a flow from $s$ to $t$ of value $\geq k$. 

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Types of Problems

Decision, Search, and Optimization

- Decision problems (example: given \( n \), is \( n \) prime?)

While using reductions and comparing problems, we typically work with the decision versions. Decision problems have Yes/No answers. This makes them easy to work with.
Types of Problems

**Decision, Search, and Optimization**

- **Decision problems** (example: given $n$, is $n$ prime?)
- **Search problems** (example: given $n$, find a factor of $n$ if it exists)
Overview

Types of Problems

Decision, Search, and Optimization

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For \textbf{Max-Flow}, the Optimization version is: Find the Maximum flow between \( s \) and \( t \). The Decision Version is: Given an integer \( k \), is there a flow of value \( \geq k \) between \( s \) and \( t \)?
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While using reductions and comparing problems, we typically work with the decision versions. Decision problems have \textbf{Yes/No} answers. This makes them easy to work with.
What is a problem, and what is an *instance* of a problem?

An instance of Bipartite Matching is a bipartite graph, and an integer $k$. The solution to this instance is "YES" if the graph has a matching of size $\geq k$, and "NO" otherwise.

You can think of a problem as a set of instances. (Formal definition in Lecture 8.)

An instance of Max-Flow is a graph $G$ with edge-capacities, two vertices $s$, $t$, and an integer $k$. The solution to this instance is "YES" if there is a flow from $s$ to $t$ of value $\geq k$, and "NO" otherwise.

What is an Algorithm for a decision Problem $X$? It takes as input an instance of $X$, and outputs either "YES" or "NO".
Problems vs Instances

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Decision Problems and Languages

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- A language $L$ is simply a subset of $\Sigma^*$; a set of strings.
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For every language $L$ there is an associated decision problem $\Pi_L$ and conversely, for every decision problem $\Pi$ there is an associated language $L_\Pi$. 
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- A language $L$ is simply a subset of $\Sigma^*$; a set of strings.

For every language $L$ there is an associated decision problem $\Pi_L$ and conversely, for every decision problem $\Pi$ there is an associated language $L_\Pi$.

- Given $L$, $\Pi_L$ is the following problem: given $x \in \Sigma^*$, is $x \in L$?
  Each string in $\Sigma^*$ is an instance of $\Pi_L$ and $L$ is the set of YES instances.
  
- Given $\Pi$ the associated language $L_\Pi = \{ I | I$ is a YES instance of $\Pi \}$.

Thus, decision problems and languages are used interchangeably.
For decision problems $X$, $Y$, a reduction from $X$ to $Y$ is:

- An algorithm ...
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(Actually, this is only one type of reduction, but this is the one we’ll use most often.)
Using reductions to solve problems

Given a reduction $\mathcal{R}$ from $X$ to $Y$, and an algorithm $A_Y$ for $Y$:

Given an instance $I_X$ of $X$, use $\mathcal{R}$ to produce an instance $I_Y$ of $Y$.
Now, use $A_Y$ to solve $I_Y$, and output the answer of $A_Y$.

In particular, if $\mathcal{R}$ and $A_Y$ are polynomial-time algorithms, $A_X$ is also polynomial-time.
Using reductions to solve problems

Given a reduction $\mathcal{R}$ from $X$ to $Y$, and an algorithm $A_Y$ for $Y$: We have an algorithm $A_X$ for $X$! Here it is:
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Using reductions to solve problems

Given a reduction $R$ from $X$ to $Y$, and an algorithm $A_Y$ for $Y$: We have an algorithm $A_X$ for $X$! Here it is:

Given an instance $I_X$ of $X$, use $R$ to produce an instance $I_Y$ of $Y$. Now, use $A_Y$ to solve $I_Y$, and output the answer of $A_Y$.

![Diagram showing reduction process]

In particular, if $R$ and $A_Y$ are polynomial-time algorithms, $A_X$ is also polynomial-time.
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Given a reduction \( \mathcal{R} \) from \( X \) to \( Y \), and an algorithm \( \mathcal{A}_Y \) for \( Y \):
We have an algorithm \( \mathcal{A}_X \) for \( X \)! Here it is:

Given an instance \( I_X \) of \( X \), use \( \mathcal{R} \) to produce an instance \( I_Y \) of \( Y \).
Now, use \( \mathcal{A}_Y \) to solve \( I_Y \), and output the answer of \( \mathcal{A}_Y \).

\[
\begin{array}{c}
I_X \xrightarrow[]{} \mathcal{R} \xrightarrow[]{} I_Y \\
\mathcal{A}_X \xrightarrow[]{} \mathcal{A}_Y \xrightarrow[]{} \text{YES} \quad \text{NO}
\end{array}
\]

In particular, if \( \mathcal{R} \) and \( \mathcal{A}_Y \) are polynomial-time algorithms, \( \mathcal{A}_X \) is also polynomial-time.
Comparing Problems

- Reductions allow us to formalize the notion of “Problem $X$ is no harder to solve than Problem $Y$”.

Bipartite Matching $\leq$ Max-Flow. Therefore, Bipartite Matching cannot be harder than Max-Flow. Equivalently, Max-Flow is at least as hard as Bipartite Matching.

More generally, if $X \leq Y$, we can say that $X$ is no harder than $Y$, or $Y$ is at least as hard as $X$. 
Comparing Problems

- Reductions allow us to formalize the notion of “Problem $X$ is no harder to solve than Problem $Y$”.
- If Problem $X$ reduces to Problem $Y$ (we write $X \leq Y$), then $X$ cannot be harder to solve than $Y$. 
  
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- **Bipartite Matching** $\leq$ **Max-Flow**. Therefore, **Bipartite Matching** cannot be harder than **Max-Flow**.
- Equivalently, **Max-Flow** is at least as hard as **Bipartite Matching**.
Comparing Problems

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- Equivalently, Max-Flow is at least as hard as Bipartite Matching.
- More generally, if $X \leq Y$, we can say that X is no harder than Y, or Y is at least as hard as X.
Part II

Examples of Reductions
Independent Sets and Cliques

Given a graph $G$, a set of vertices $V'$ is:
Independent Sets and Cliques

Given a graph $G$, a set of vertices $V'$ is:

- An **independent set** if no two vertices of $V'$ are connected by an edge of $G$. 
Given a graph $G$, a set of vertices $V'$ is:

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![Graph diagram]
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![Diagram of a graph with vertices and edges]
Independent Sets and Cliques

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- An **independent set** if no two vertices of $V'$ are connected by an edge of $G$.
- A **clique** if *every* pair of vertices in $V'$ is connected by an edge of $G$.
The **Independent Set** and **Clique** Problems

**The Independent Set Problem:**

**Input** A graph $G$ and an integer $k$.

**Goal** Decide whether $G$ has an independent set of size $\geq k$. 

The **Independent Set** and **Clique** Problems

**The Independent Set Problem:**

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**The Clique Problem:**

**Input** A graph $G$ and an integer $k$.

**Goal** Decide whether $G$ has a clique of size $\geq k$. 
Recall

For decision problems $X, Y$, a reduction from $X$ to $Y$ is:

- An algorithm ...
For decision problems $X$, $Y$, a reduction from $X$ to $Y$ is:

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- such that the solution (YES/NO) to $I_Y$ is the same as the solution to $I_X$. 
Reducing **Independent Set** to **Clique**

An instance of **Independent Set** is a graph $G$ and an integer $k$. We can convert $G$ to $\bar{G}$, in which $(u, v)$ is an edge iff $(u, v)$ is not an edge of $G$. ($\bar{G}$ is the complement of $G$.) We use $\bar{G}$ and $k$ as the instance of **Clique**.
Reducing **INDEPENDENT SET** to **CLIQUE**

An instance of **INDEPENDENT SET** is a graph $G$ and an integer $k$. 

![Graph](image-url)
Reducing **INDEPENDENT SET** to **CLIQUE**

An instance of **INDEPENDENT SET** is a graph $G$ and an integer $k$.

Convert $G$ to $\overline{G}$, in which $(u, v)$ is an edge iff $(u, v)$ is **not** an edge of $G$. ($\overline{G}$ is the *complement* of $G$.)

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\[\text{Diagram of } \overline{G}\]
Reducing **INDEPENDENT SET** to **CLIQUE**

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We showed that \textbf{Independent Set} $\leq$ \textbf{Clique}.

What does this mean?
We showed that \textbf{Independent Set} $\leq$ \textbf{Clique}. What does this mean?

If we have an algorithm for \textbf{Clique}, we have an algorithm for \textbf{Independent Set}. 
We showed that \textbf{Independent Set} $\leq$ \textbf{Clique}.

What does this mean?

If we have an algorithm for \textbf{Clique}, we have an algorithm for \textbf{Independent Set}.

The \textbf{Clique} Problem is \textit{at least as hard as} the \textbf{Independent Set} problem.
DFAs and NFAs

DFAs (Remember 273?) are automata that accept regular languages. NFAs are the same, except that they are non-deterministic, while DFAs are deterministic.
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Every NFA can be converted to a DFA that accepts the same language using the subset construction.

(How long does this take?)
DFAs and NFAs

DFAs (Remember 273?) are automata that accept regular languages. NFAs are the same, except that they are non-deterministic, while DFAs are deterministic.

Every NFA can be converted to a DFA that accepts the same language using the subset construction.

(How long does this take?)
The smallest DFA equivalent to an NFA with $n$ states may have $\approx 2^n$ states.
A DFA $M$ is said to be universal if it accepts every string. That is, $L(M) = \Sigma^*$, the set of all strings.
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The DFA Universality Problem:

Input  A DFA $M$

Goal    Decide whether $M$ is universal.
DFA Universality

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The DFA \textsc{Universality} Problem:

\textbf{Input} A DFA $M$

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How do we solve \textsc{DFA Universality}?
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The DFA Universality Problem:

**Input**  A DFA $M$

**Goal**  Decide whether $M$ is universal.

How do we solve DFA Universality?
We check if $M$ has any reachable non-final state.
Alternatively, minimize $M$ to obtain $M'$ and see if $M'$ has a single state which is an accepting state.
An NFA $N$ is said to be universal if it accepts every string. That is, $L(N) = \Sigma^*$, the set of all strings.

The NFA Universality Problem:

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**Goal**  Decide whether $N$ is universal.

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Reduce it to DFA Universality?
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Goal  Decide whether $N$ is universal.

How do we solve NFA Universality?
Reduce it to DFA Universality?
Given an NFA $N$, convert it to an equivalent DFA $M$, and use the DFA Universality Algorithm.
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The NFA Universality Problem:
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Goal Decide whether $N$ is universal.

How do we solve NFA Universality?
Reduce it to DFA Universality?
Given an NFA $N$, convert it to an equivalent DFA $M$, and use the DFA Universality Algorithm.

The reduction takes exponential time!
We say that an algorithm is efficient if it runs in polynomial-time.
Polynomial-time reductions

We say that an algorithm is **efficient** if it runs in polynomial-time.

To find efficient algorithms for problems, we are only interested in **polynomial-time** reductions. Reductions that take longer are not useful.
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If we have a polynomial-time reduction from problem $X$ to problem $Y$ (we write $X \leq_P Y$), and a poly-time algorithm $A_Y$ for $Y$, we have a polynomial-time/efficient algorithm for $X$. 
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![Diagram](attachment:image.png)
Polynomial-time reductions and hardness

For decision problems $X$ and $Y$, if $X \leq_P Y$, and $Y$ has an efficient algorithm, $X$ has an efficient algorithm.
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For decision problems $X$ and $Y$, if $X \leq_P Y$, and $Y$ has an efficient algorithm, $X$ has an efficient algorithm.

If you believe that Independent Set does not have an efficient algorithm, why should you believe the same of Clique?
Polynomial-time reductions and hardness

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If you believe that $\text{INDEPENDENT SET}$ does not have an efficient algorithm, why should you believe the same of $\text{CLIQUE}$?

Because we showed $\text{INDEPENDENT SET} \leq_P \text{CLIQUE}$. If $\text{CLIQUE}$ had an efficient algorithm, so would $\text{INDEPENDENT SET}$!
Polynomial-time reductions and hardness

For decision problems $X$ and $Y$, if $X \leq_P Y$, and $Y$ has an efficient algorithm, $X$ has an efficient algorithm.

If you believe that Independent Set does not have an efficient algorithm, why should you believe the same of Clique?

Because we showed Independent Set $\leq_P$ Clique. If Clique had an efficient algorithm, so would Independent Set!

If $X \leq_P Y$ and $X$ does not have an efficient algorithm, $Y$ cannot have an efficient algorithm!
Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from $X$ to $Y$. Then for any instance $I_X$ of $X$, the size of the instance $I_Y$ of $Y$ produced from $I_X$ by $\mathcal{R}$ is polynomial in the size of $I_X$. 

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input $I_X$ of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$. $I_Y$ is the output of $\mathcal{R}$ on input $I_X$. $\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. 

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.
Proposition

Let \( \mathcal{R} \) be a polynomial-time reduction from \( X \) to \( Y \). Then for any instance \( I_X \) of \( X \), the size of the instance \( I_Y \) of \( Y \) produced from \( I_X \) by \( \mathcal{R} \) is polynomial in the size of \( I_X \).

Proof.

\( \mathcal{R} \) is a polynomial-time algorithm and hence on input \( I_X \) of size \( |I_X| \) it runs in time \( p(|I_X|) \) for some polynomial \( p() \).

\( I_Y \) is the output of \( \mathcal{R} \) on input \( I_X \)

\( \mathcal{R} \) can write at most \( p(|I_X|) \) bits and hence \( |I_Y| \leq p(|I_X|) \).
Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from $X$ to $Y$. Then for any instance $I_X$ of $X$, the size of the instance $I_Y$ of $Y$ produced from $I_X$ by $\mathcal{R}$ is polynomial in the size of $I_X$.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input $I_X$ of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$. $I_Y$ is the output of $\mathcal{R}$ on input $I_X$. $\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. 

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.
Proposition

\[ X \leq_P Y \text{ and } Y \leq_P Z \text{ implies that } X \leq_P Z. \]

Note: \( X \leq_P Y \) does not imply that \( Y \leq_P X \) and hence it is very important to know the FROM and TO in a reduction.

To prove \( X \leq_P Y \) you need to show a reduction FROM \( X \) TO \( Y \) in other words show that an algorithm for \( Y \) implies an algorithm for \( X \).
Vertex Cover

Given a graph $G = (V, E)$, a set of vertices $S$ is:
Vertex Cover

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- A **vertex cover** if every $e \in E$ has at least one endpoint in $S$. 
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The **Vertex Cover** Problem:

**Input**  A graph $G$ and integer $k$

**Goal**  Decide whether there is a vertex cover of size $\leq k$
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Can we relate **Independent Set** and **Vertex Cover**?
Relationship between Vertex Cover and Independent Set

**Proposition**

Let $G = (V, E)$ be a graph. $S$ is an independent set if and only if $V \setminus S$ is a vertex cover.
Relationship between Vertex Cover and Independent Set

Proposition

Let \( G = (V, E) \) be a graph. \( S \) is an independent set if and only if \( V \setminus S \) is a vertex cover.

Proof.

(\( \Rightarrow \)) Let \( S \) be an independent set.
Consider any edge \((u, v) \in E\).
Since \( S \) is an independent set, either \( u \not\in S \) or \( v \not\in S \).
Thus, either \( u \in V \setminus S \) or \( v \in V \setminus S \).
\( V \setminus S \) is a vertex cover.

(\( \Leftarrow \)) Let \( V \setminus S \) be some vertex cover.
Consider \( u, v \in S \).
\((u, v) \) is not an edge, as otherwise \( V \setminus S \) does not cover \((u, v) \).
\( S \) is thus an independent set.
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Let $G = (V, E)$ be a graph. $S$ is an independent set if and only if $V \setminus S$ is a vertex cover.

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(⇒) Let $S$ be an independent set
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$(\Rightarrow)$ Let $S$ be an independent set

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Let $G$, a graph with $n$ vertices, and an integer $k$ be an instance of the \textsc{Independent Set} problem.
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$G$ has an independent set of size $\geq k$ iff $G$ has a vertex cover of size $\leq n - k$
**Independent Set** $\leq_P$ **Vertex Cover**

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$(G, k)$ is an instance of **Independent Set**, and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
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$(G, k)$ is an instance of Independent Set, and $(G, n - k)$ is an instance of Vertex Cover with the same answer.

Therefore, Independent Set $\leq_P$ Vertex Cover. Also Vertex Cover $\leq_P$ Independent Set.
Suppose you work for the United Nations. Let $U$ be the set of all languages spoken by people across the world. The United Nations also has a set of translators, all of whom speak English, and some other languages from $U$. 

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More General problem: Find/Hire a small group of people who can accomplish a large number of tasks.
The **Set Cover** Problem

**Input**  Given a set $U$ of $n$ elements, a collection $S_1, S_2, \ldots S_m$ of subsets of $U$, and an integer $k$

**Goal**  Is there is a collection of at most $k$ of these sets $S_i$ whose union is equal to $U$?
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**Example**

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$ with

$S_1 = \{3, 7\}$  $S_2 = \{3, 4, 5\}$
$S_3 = \{1\}$  $S_4 = \{2, 4\}$
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\]

\( \{S_2, S_6\} \) is a set cover
**Vertex Cover \( \leq_P \) Set Cover**

Given graph \( G = (V, E) \) and integer \( k \) as instance of **Vertex Cover**, construct an instance of **Set Cover** as follows:

- Number \( k \) for the **Set Cover** instance is the same as the number \( k \) given for the **Vertex Cover** instance.
- \( U = E \)

We will have one set corresponding to each vertex; \( S_v = \{ e | e \text{ is incident on } v \} \)

Observe that \( G \) has vertex cover of size \( k \) if and only if \( U, \{ S_v | v \in V \} \) has a set cover of size \( k \).

(Exercise: Prove this.)
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**Vertex Cover** $\leq_P$ **Set Cover**: Example

Let $U = \{a, b, c, d, e, f, g\}$, $k = 2$ with $S_1 = \{c, g\}$, $S_2 = \{b, d\}$, $S_3 = \{c, d, e\}$, $S_4 = \{e, f\}$, $S_5 = \{a\}$, $S_6 = \{a, b, f, g\}$.

$\{S_3, S_6\}$ is a set cover.
**Vertex Cover \( \leq_p \) Set Cover: Example**

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S_1 = \{c, g\} \quad S_2 = \{b, d\} \\
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- \( S_6 = \{a, b, f, g\} \)

\( \{S_3, S_6\} \) is a set cover

\( \{3, 6\} \) is a vertex cover
**Integer Linear Programming**

**Problem**

Find a vector \( x \in \mathbb{Z}^d \) (integer values) that

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{d} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for } i = 1 \ldots n
\end{align*}
\]

Input is matrix \( A = (a_{ij}) \in \mathbb{Q}^{n \times d} \), column vector \( b = (b_i) \in \mathbb{Q}^n \), and row vector \( c = (c_j) \in \mathbb{Q}^d \) where \( \mathbb{Q} \) is the set of rational numbers.

**Decision version:** Given \( A, b, c \) and \( \nu \in \mathbb{Q} \), is the optimum of the integer program at least \( \nu \)?
**Independent Set \leq_p ILP**

\( G = (V, E) \) and \( k \) are an instance of Independent Set. How do we reduce to an instance of ILP?
**Independent Set \leq_p ILP**

$G = (V, E)$ and $k$ are an instance of **Independent Set**. How do we reduce to an instance of ILP?

- For each vertex $v$ a variable $x_v$. Enforce that $x_v \in \{0, 1\}$: 1 means $v$ chosen in independent set, 0 otherwise.
- For each edge $(u, v)$ can only pick $u$ or $v$. 

Opt value of ILP instance is $k$ iff $G$ has indep set of size $k$. 
Independent Set $\leq_P$ ILP

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- For each edge $(u, v)$ can only pick $u$ or $v$.

$$\max \sum_{v \in V} x_v$$
$$x_u + x_v \leq 1 \quad \text{for each} \quad (u, v) \in E$$
$$0 \leq x_v \leq 1 \quad \text{for each} \quad v \in V$$

$x_v$ is an integer for each $v \in V$

Opt value of ILP instance is $k$ iff $G$ has indep set of size $k$
Instance of \textbf{Set Cover}: set $U$ of $n$ elements and subsets $S_1, S_2, \ldots, S_m$ of $U$ and integer $k$. Reduction to ILP?
**Set Cover \(\leq_p\) ILP**

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- For each set \(S_i\) a variable \(x_i\). Enforce that \(x_i \in \{0, 1\}\): 1 means \(i\) chosen in cover 0 otherwise

- For each \(j \in U\), at least one set containing \(j\) in cover.

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**Set Cover \leq_p ILP**

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- For each set $S_i$ a variable $x_i$. Enforce that $x_i \in \{0, 1\}$: 1 means $i$ chosen in cover 0 otherwise.
- For each $j \in U$, at least one set containing $j$ in cover.

\[
\min \sum_{v \in V} x_i \\
\sum_{i: j \in S_i} x_i \geq 1 \quad 1 \leq j \leq n
\]

\[
0 \leq x_i \leq 1 \quad 1 \leq i \leq m
\]

$x_i$ is an integer for $1 \leq i \leq m$

Opt value of ILP instance is $k$ iff there is a set cover of size $k$. 
Summary

We looked at polynomial-time reductions.
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**Using polynomial-time reductions**

- If $X \leq_P Y$, and we have an efficient algorithm for $Y$, we have an efficient algorithm for $X$. 
We looked at polynomial-time reductions.

**Using polynomial-time reductions**

- If $X \leq_P Y$, and we have an efficient algorithm for $Y$, we have an efficient algorithm for $X$.
- If $X \leq_P Y$, and there is no efficient algorithm for $X$, there is no efficient algorithm for $Y$. 
Summary

We looked at polynomial-time reductions.

Using polynomial-time reductions

- If $X \leq_P Y$, and we have an efficient algorithm for $Y$, we have an efficient algorithm for $X$.
- If $X \leq_P Y$, and there is no efficient algorithm for $X$, there is no efficient algorithm for $Y$.

We looked at some examples of reductions between \textsc{Independent Set}, \textsc{Clique}, \textsc{Vertex Cover}, \textsc{Set Cover}, ILP etc.