CS 473: Algorithms

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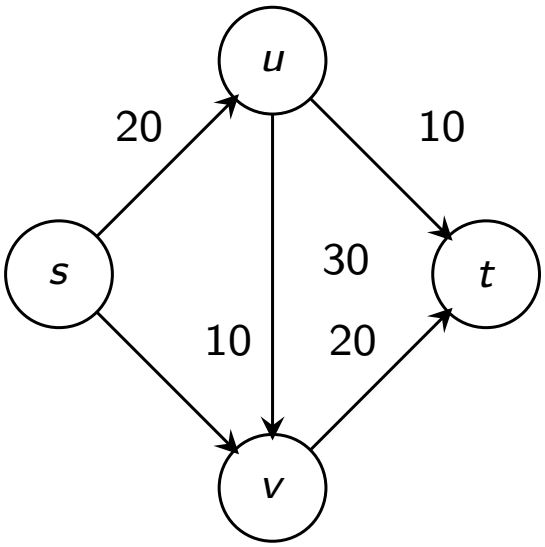
Part I

Algorithm(s) for Maximum Flow

Greedy Approach



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2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$
3. *Augment* flow along this path
4. Repeat augmentation for as long as possible.

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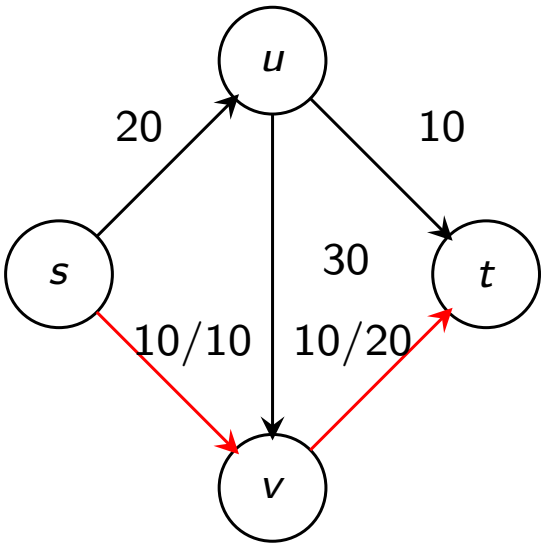

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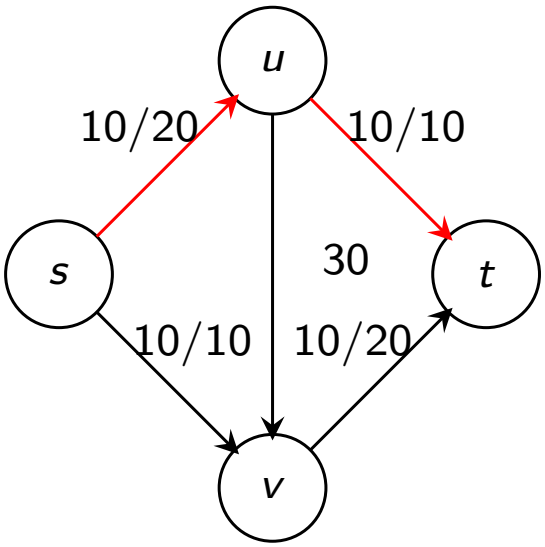

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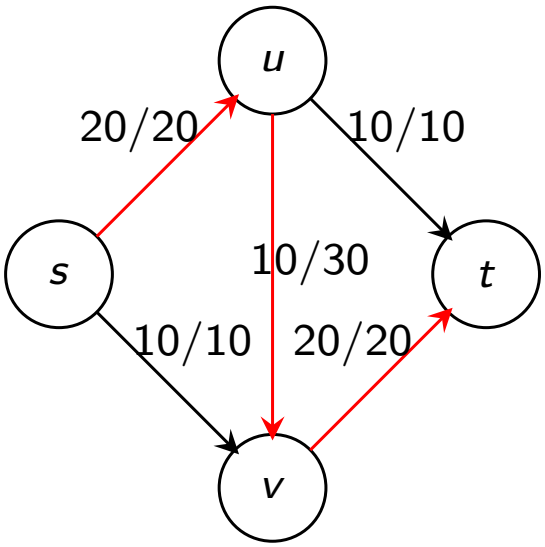

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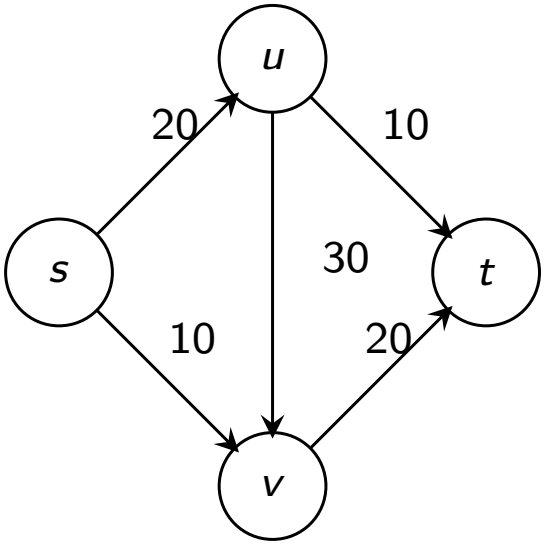

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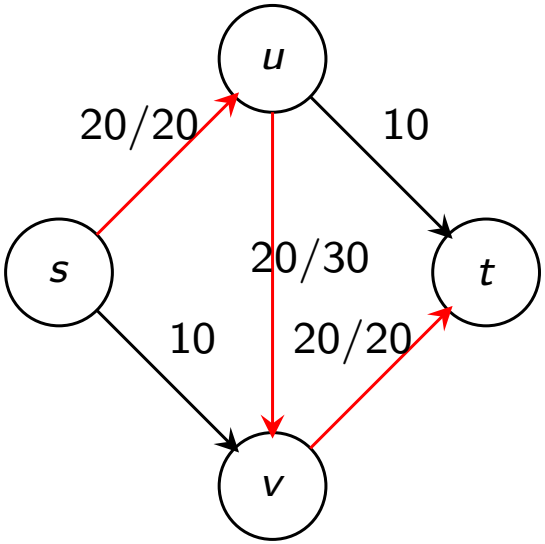

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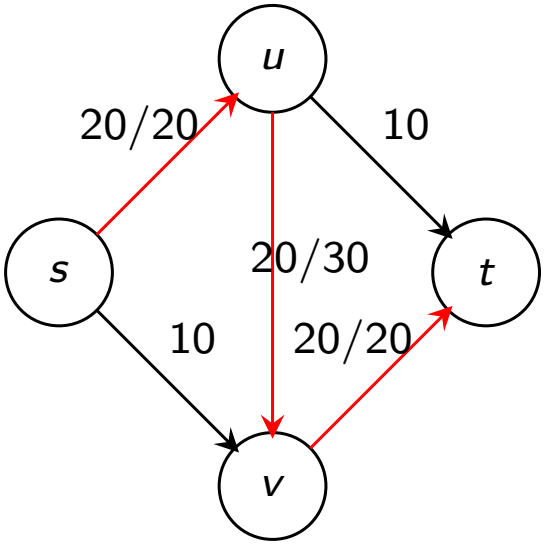

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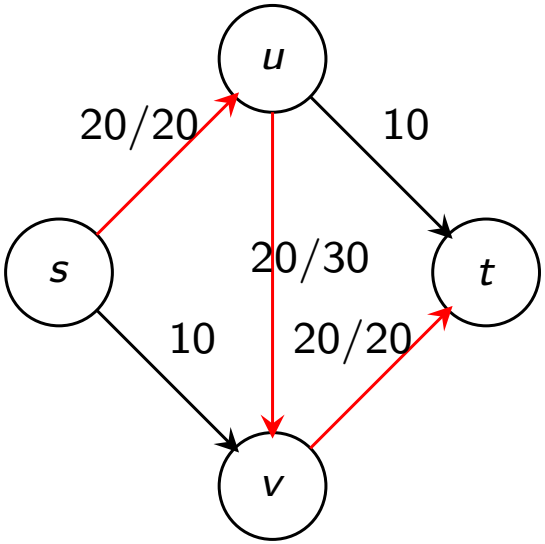

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Need to "push-back" flow along edge (u, v)

Residual Graph

Definition

For a network $G = (V, E)$ and flow f, the residual graph $G_f = (V', E')$ of G with respect to f is

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- Backward Edges: For each edge $e = (u, v) \in E$ with $f(e) > 0$, we $(v, u) \in E'$ with capacity $f(e)$

Residual Graph Example

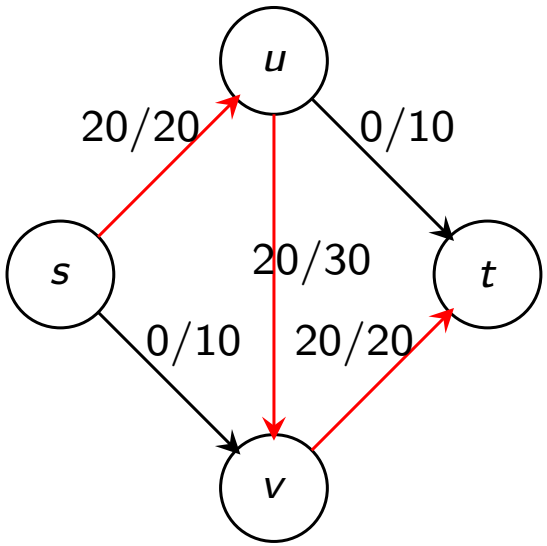

Figure: Flow in red edges

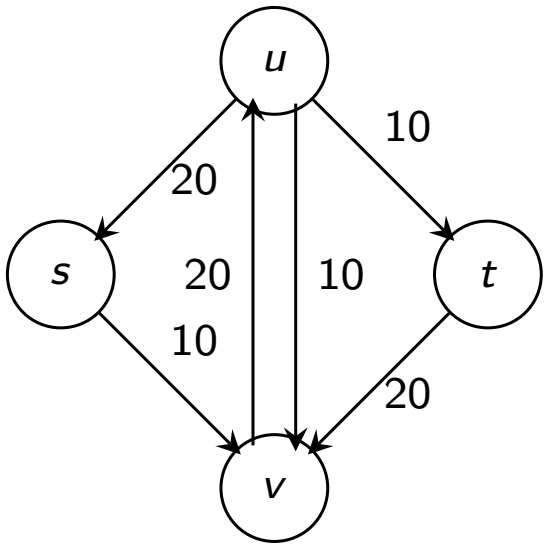

Figure: Residual Graph

Ford-Fulkerson Algorithm

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for every edge e, f(e) = 0
G_f is residual graph of G with respect to f
while G_f has a simple s-t path
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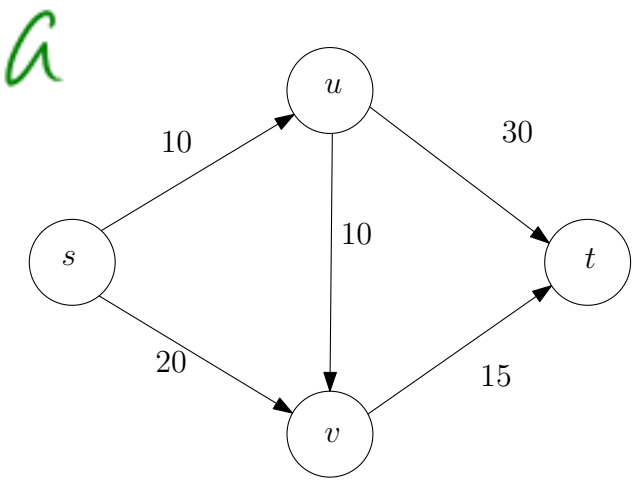
augment(f,P)
    let b be bottleneck capacity, i.e., min capacity of edges in P
    for each edge (u,v) in P
        if e=(u,v) is a forward edge
            f(e) = f(e) + b
        else (* (u,v) is a backward edge *)
            let e = (v,u) (* (v,u) is in G *)
            f(e) = f(e) - b
    return f
```

Example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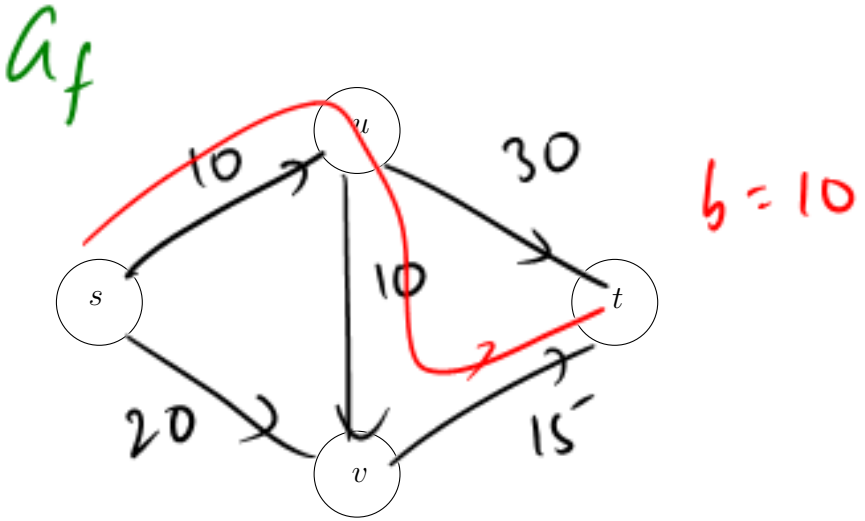
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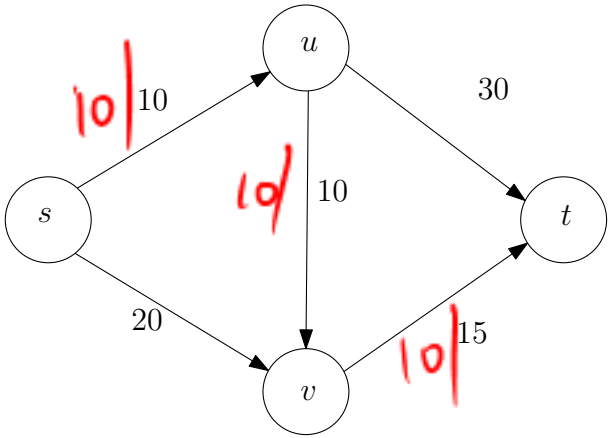
b = 10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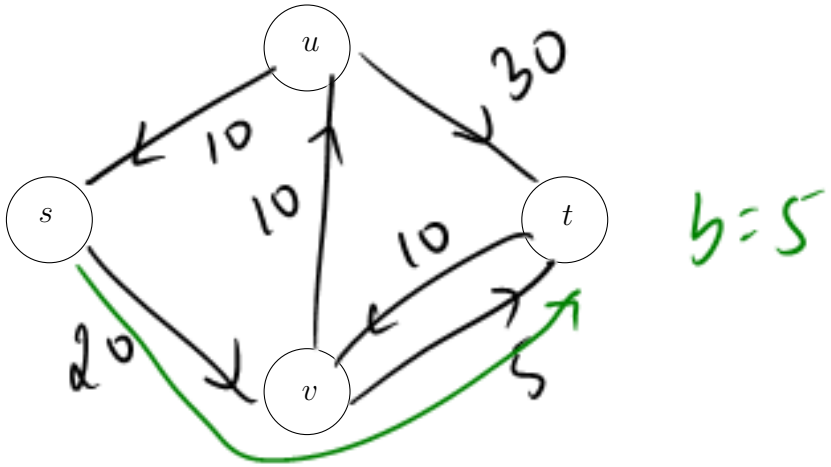
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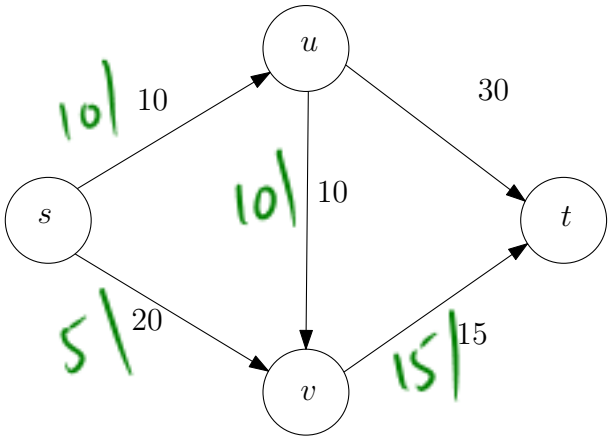
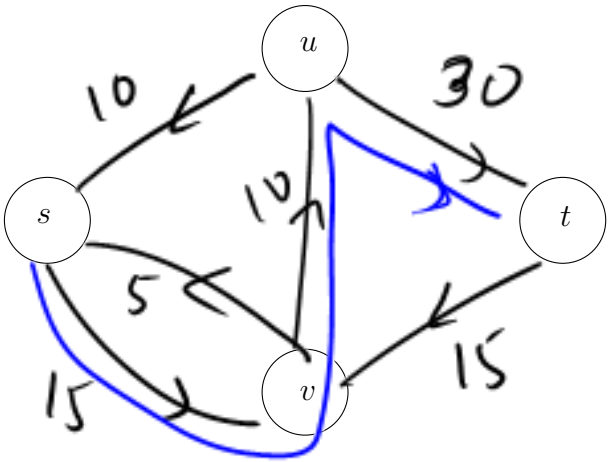
Example continued





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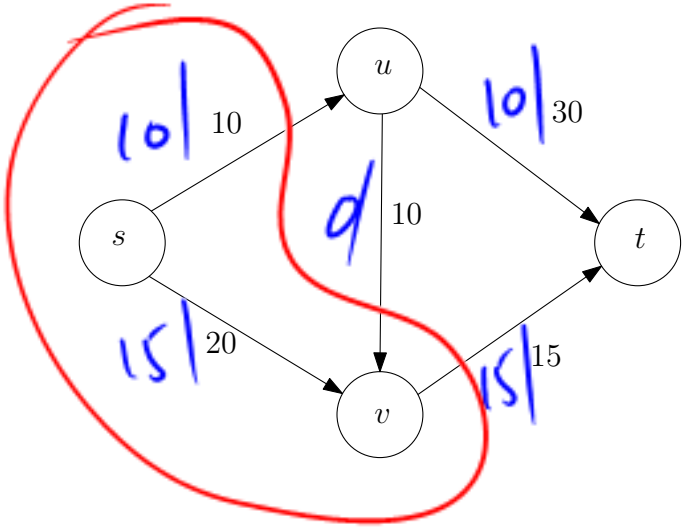

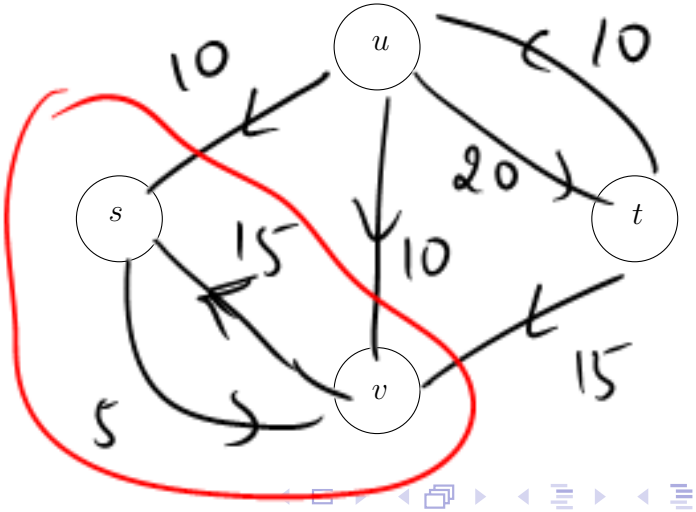

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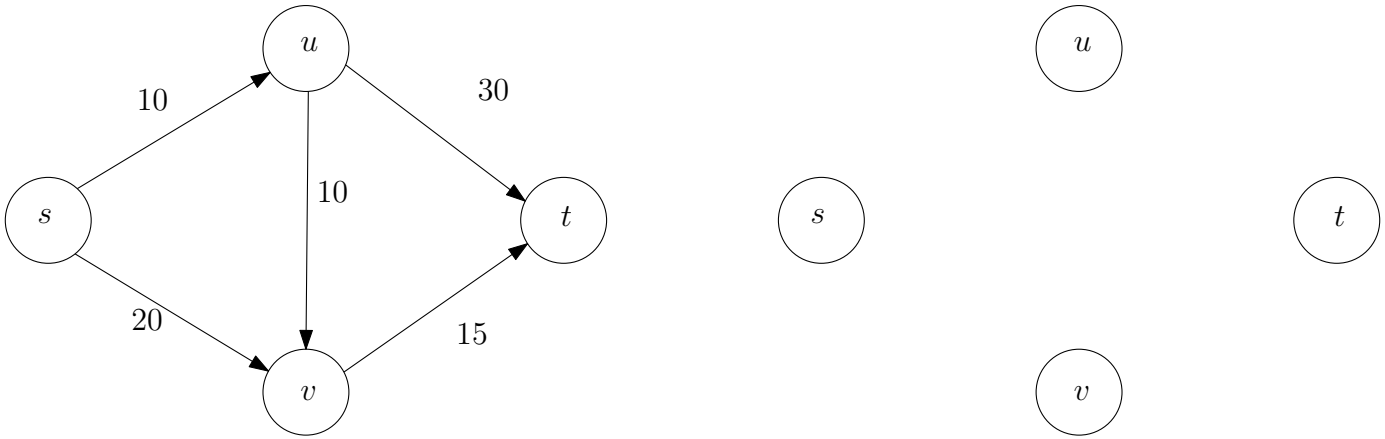

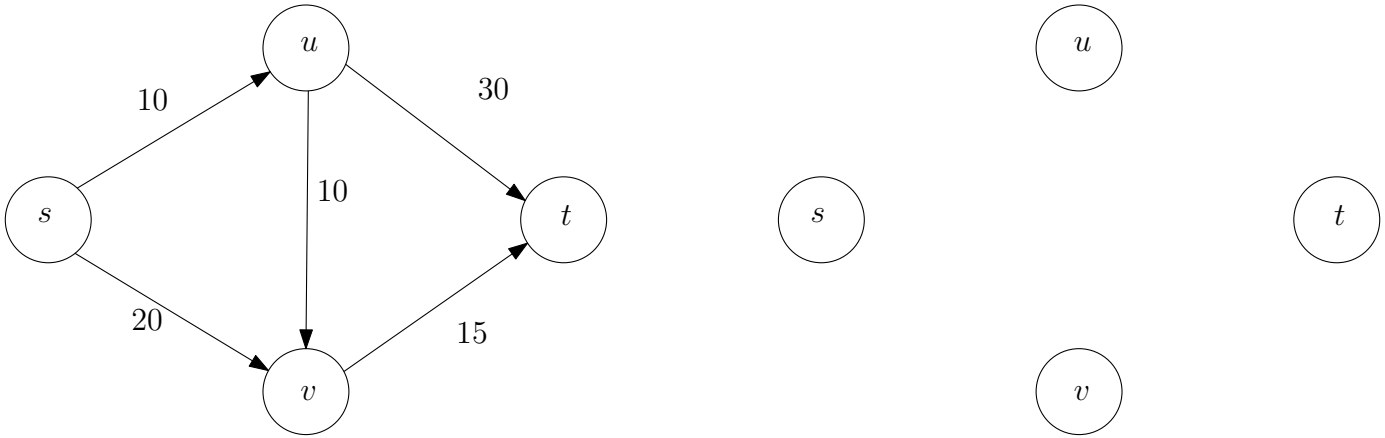

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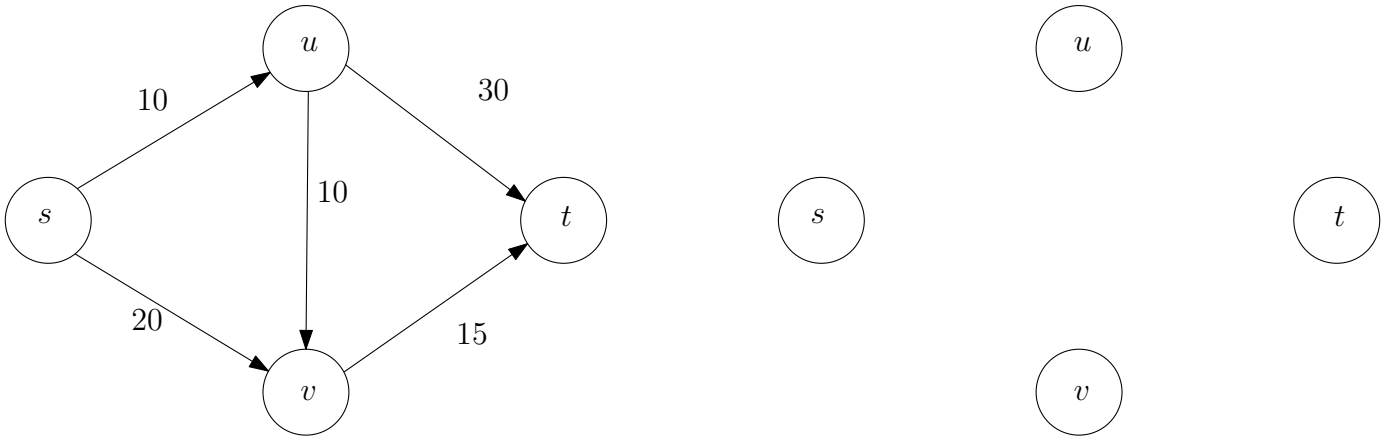

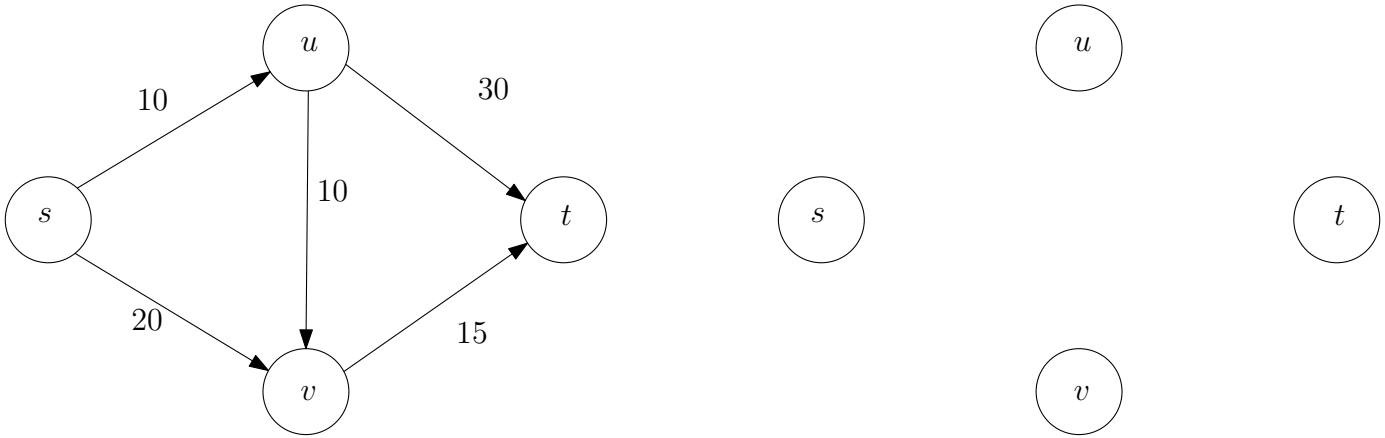

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If f is a flow and P is a simple s-t path in G_f, then $f' = \texttt{augment}(f, P)$ is also a flow.

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- Conservation constraint: Let v be an internal node. Let e_1, e_2 be edges of P incident to v. Four cases based on whether e_1, e_2 are forward or backward edges. Check cases (see fig next slide). □

Properties about Augmentation: Conservation Constraint

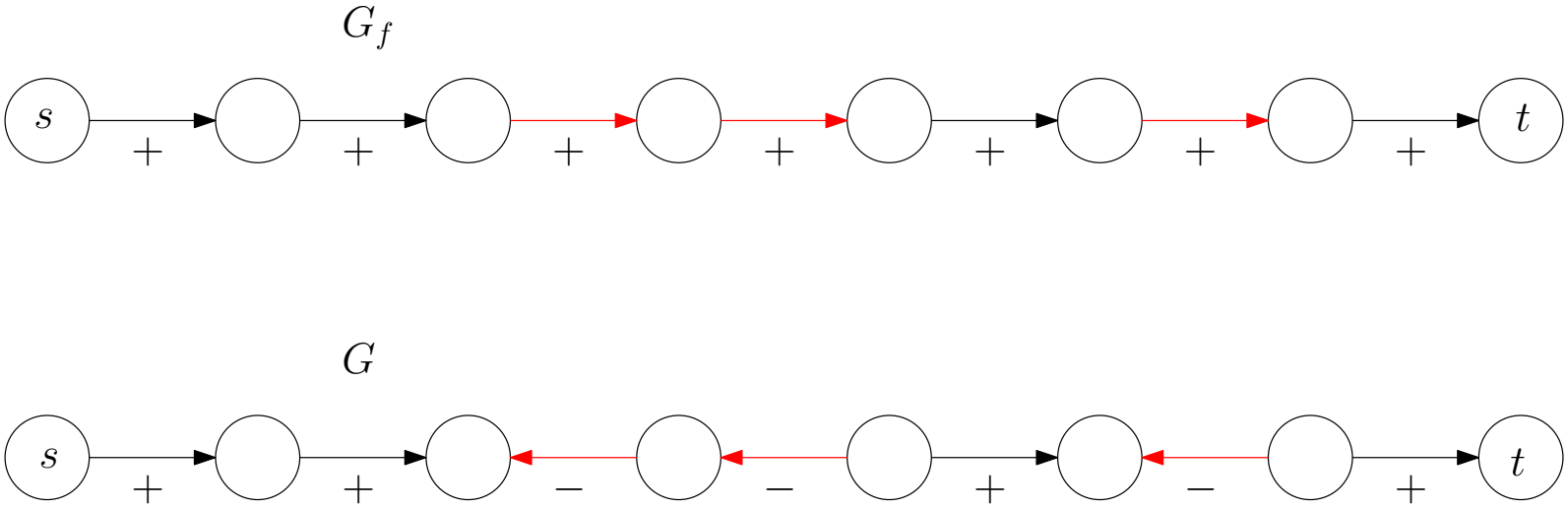

Figure: Augmenting path P in G_f and corresponding change of flow in G. Red edges are backward edges.

Properties about Augmentation: Integer Flow

Lemma

At every stage of the Ford-Fulkerson algorithm, the flow values $f(e)$ and the residual capacities in G_f are integers

Proof.

Initial flow and residual capacities are integers. Suppose lemma holds for j iterations. Then in $j+1$st iteration, minimum capacity edge b is an integer, and so flow after augmentation is an integer. □

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- P is simple and so never returns to s
- Thus, value of flow increases by the flow on edge e □

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- Running time is $O(C(n + m))$ (or $O(mC)$).

Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the running time be as $\Omega(mC)$ or is our analysis weak?

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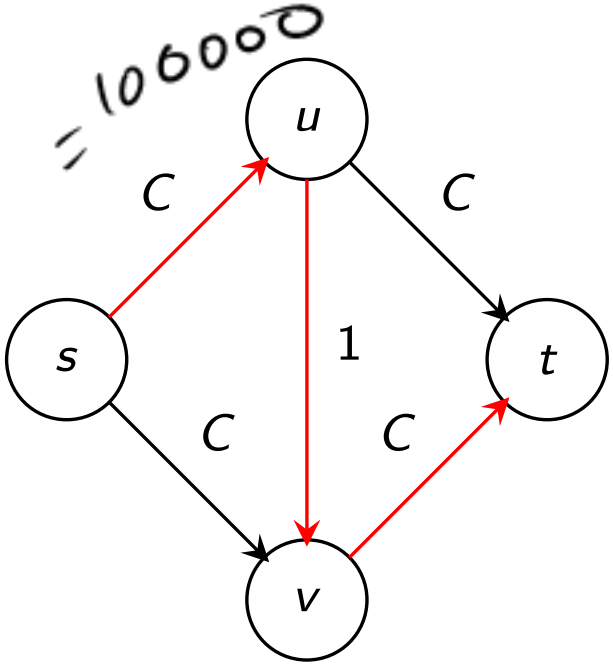

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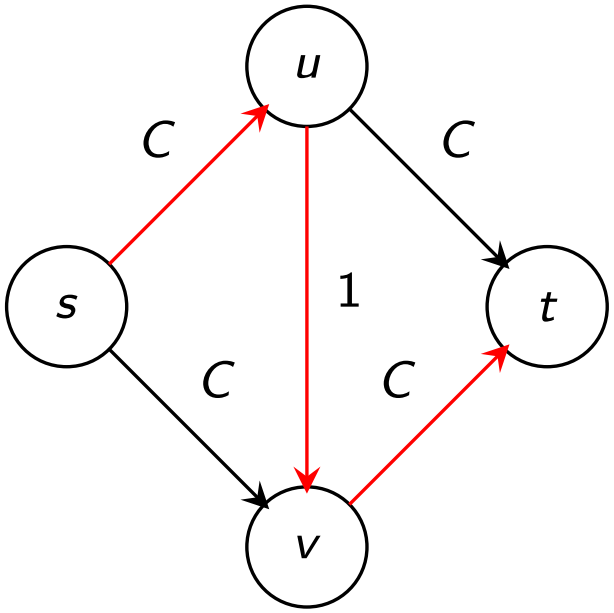

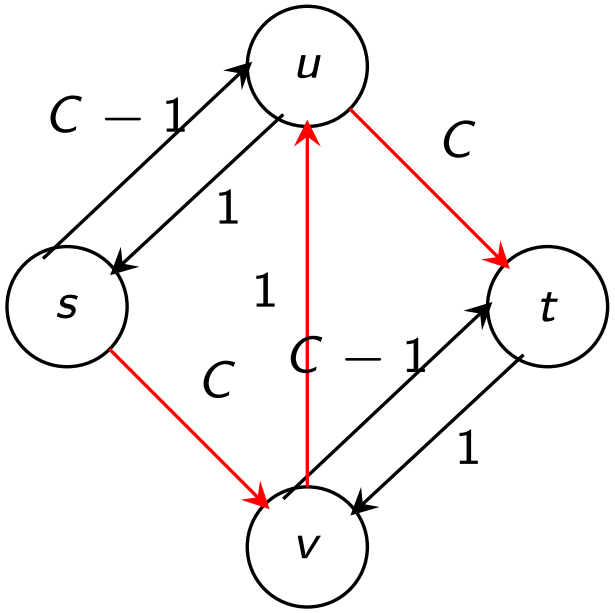

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Question: When the algorithm terminates, is the flow computed the maximum s-t flow?

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Proof idea: show a cut of value equal to the flow. Also shows that maximum flow is equal to minimum cut!

Recalling Cuts

Definition

Given a flow network an s-t cut is a set of edges $E' \subset E$ such that removing E' *disconnects* s from t: in other words there is no directed $s \rightarrow t$ path in $E - E'$. *Capacity* of cut E' is $\sum_{e \in E'} c(e)$.

Let $A \subset V$ such that

- $s \in A$, $t \notin A$
- $B = V - A$ and hence $t \in B$

Define $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$

Claim

(A, B) is an s-t cut.

Recall: Every *minimal* s-t cut E' is a cut of the form (A, B).

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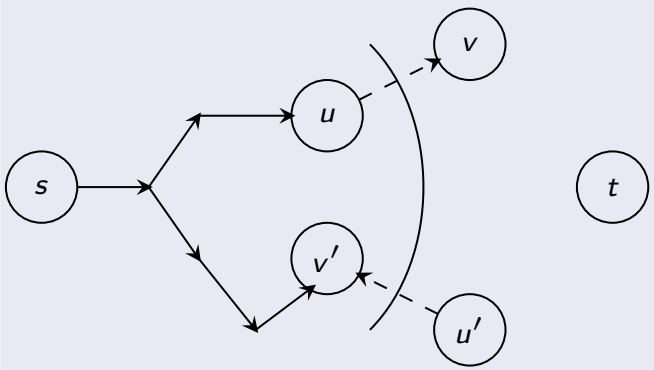

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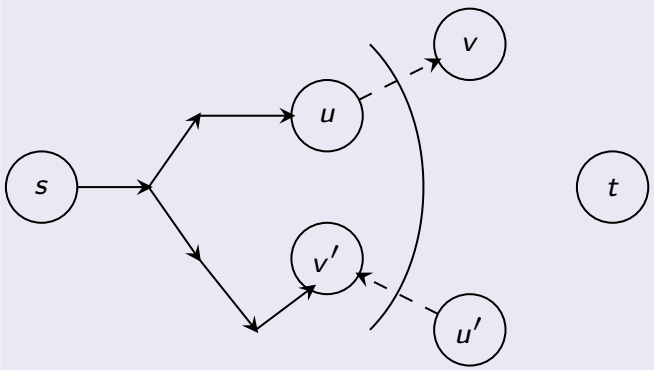

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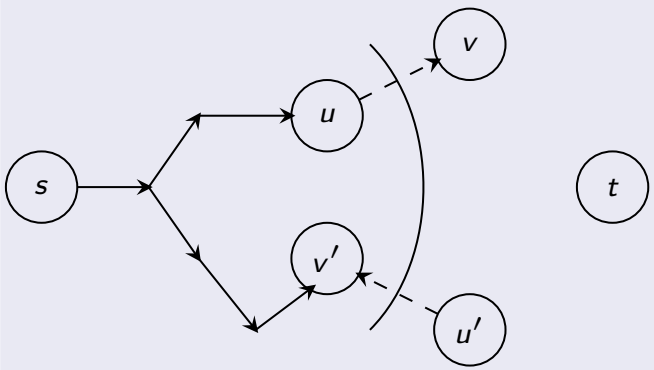
If there is no s-t path in G_f then there is some cut (A, B) such that $v(f) = c(A, B)$

Proof.

Let A be all vertices reachable from s in G_f; $B = V \setminus A$



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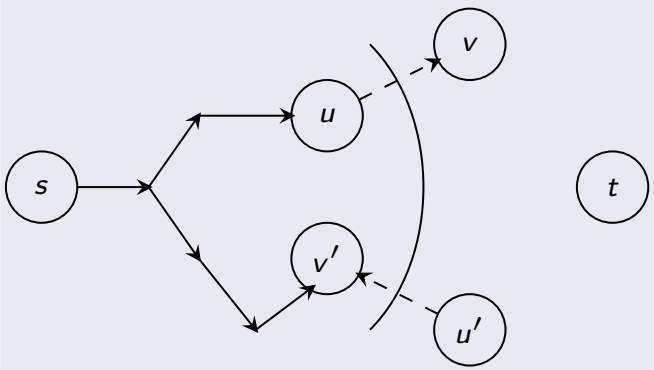

- $s \in A$ and $t \in B$. So (A, B) is an s-t cut in G
- If $e = (u, v) \in G$ with $u \in A$ and $v \in B$, then $f(e) = c(e)$ (saturated edge) because otherwise v is reachable from s in G

Lemma Proof Continued

Proof.

□

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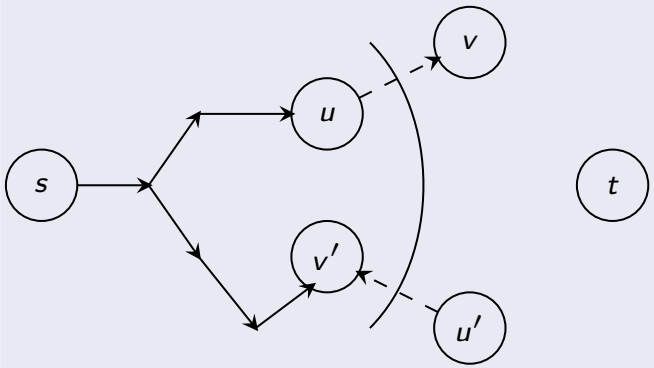

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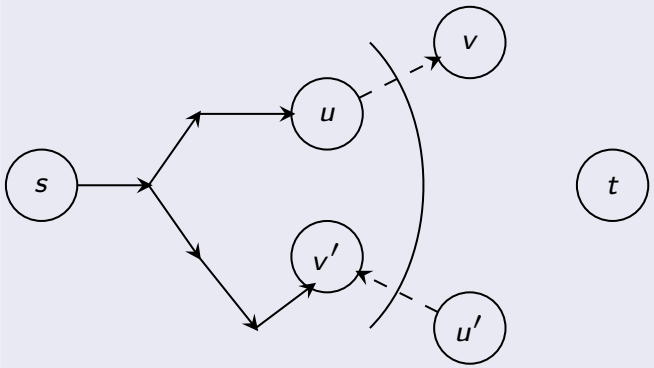

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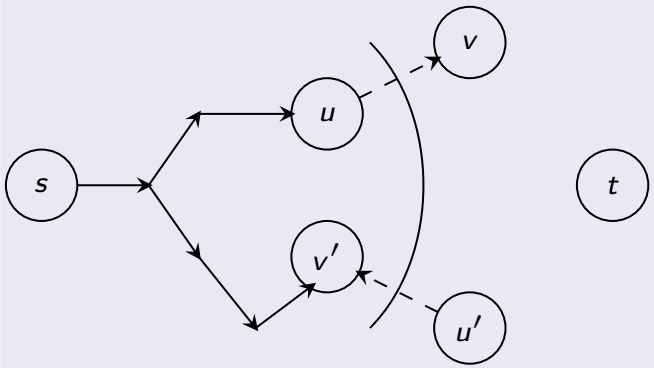

- If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s

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Lemma Proof Continued

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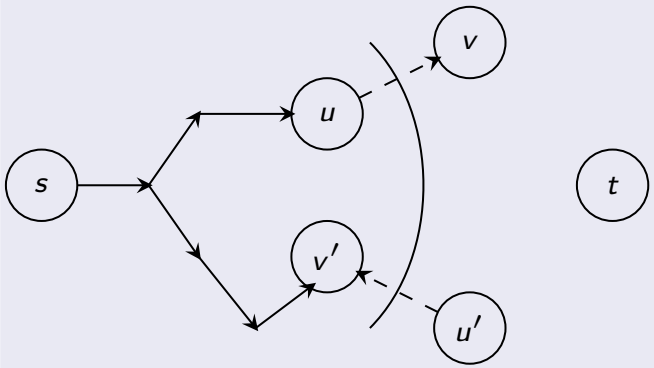

- If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s
- Thus,

$$\begin{aligned} v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\ &= f^{\text{out}}(A) - 0 \\ &= c(A, B) - 0 \\ &= c(A, B) \end{aligned}$$

□

Example

Ford-Fulkerson Correctness

Theorem

The flow returned by the algorithm is the maximum flow.

Ford-Fulkerson Correctness

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Proof.

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Ford-Fulkerson Correctness

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- For any flow f and s-t cut (A, B), $v(f) \leq c(A, B)$

$\square$

Ford-Fulkerson Correctness

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The flow returned by the algorithm is the maximum flow.

Proof.

- For any flow f and s-t cut (A, B), $v(f) \leq c(A, B)$
- For flow f^* returned by algorithm, $v(f^*) = c(A^*, B^*)$ for some s-T cut (A^*, B^*)

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- Hence, f^* is maximum

□

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network G, the value of a maximum s-t flow is equal to the capacity of the minimum s-t cut.

Proof.

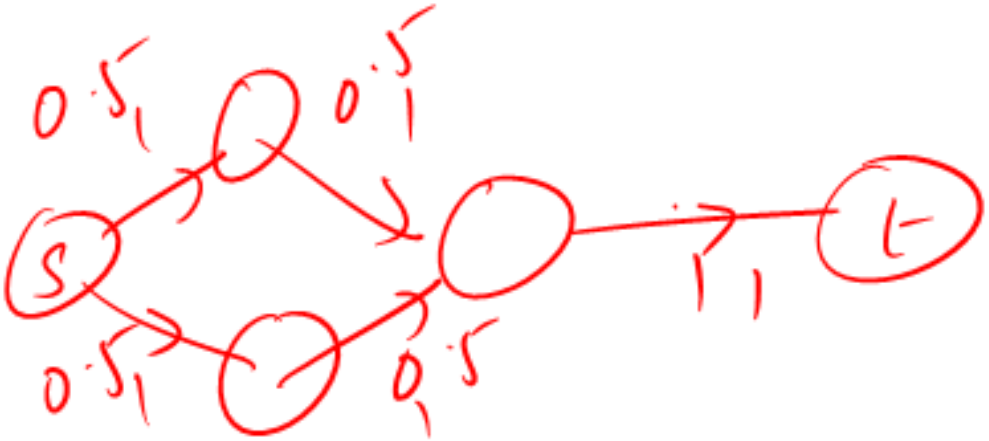
Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. □

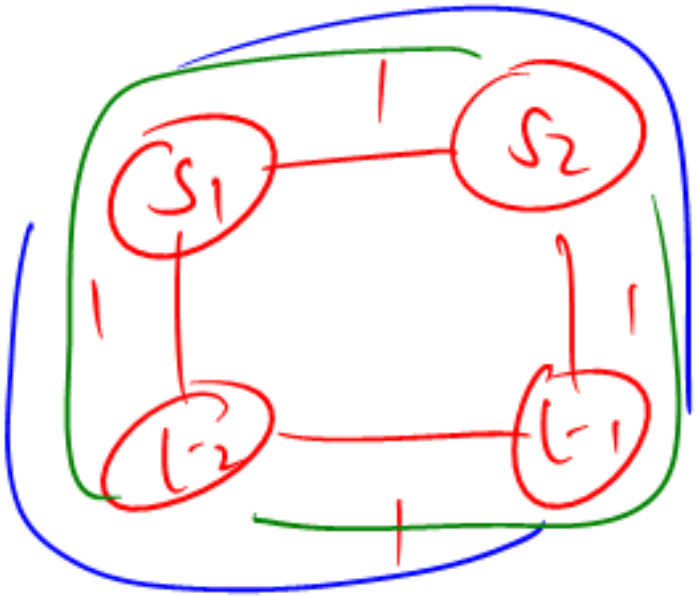
Theorem

For any network G with integer capacities, there is a maximum s-t flow that is integer valued.

Proof.

Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. □


0.5
1
0.5
1
s
t
1
1
0.5
1
0.5
1


1
s1
s2
1
1
t2
t1
1

Efficiency of Ford-Fulkerson

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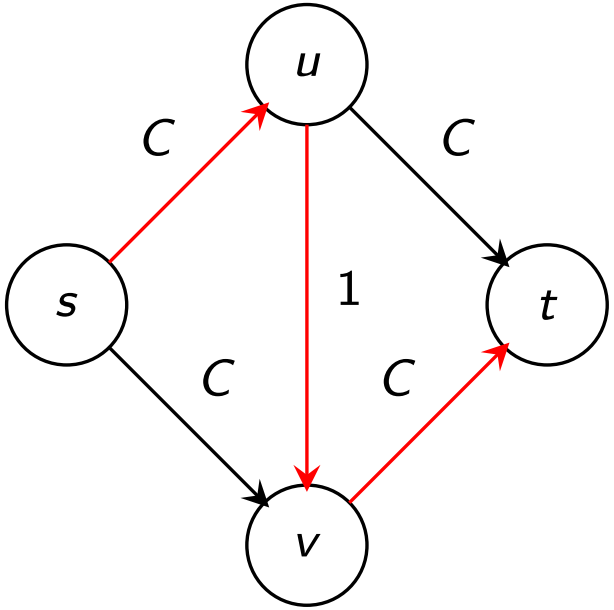

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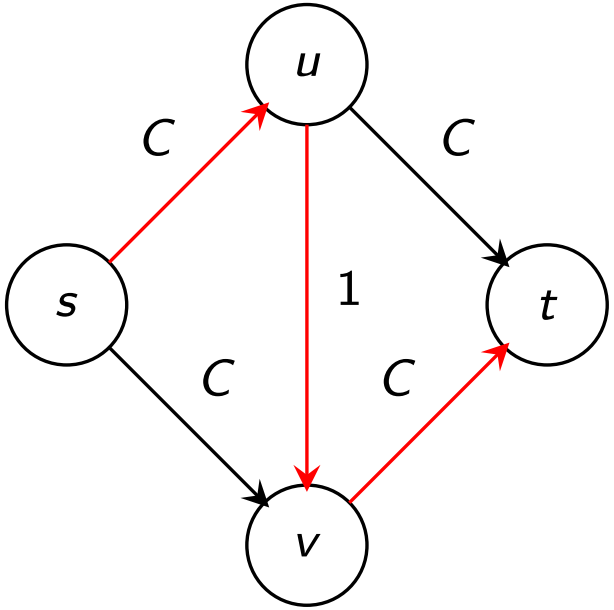

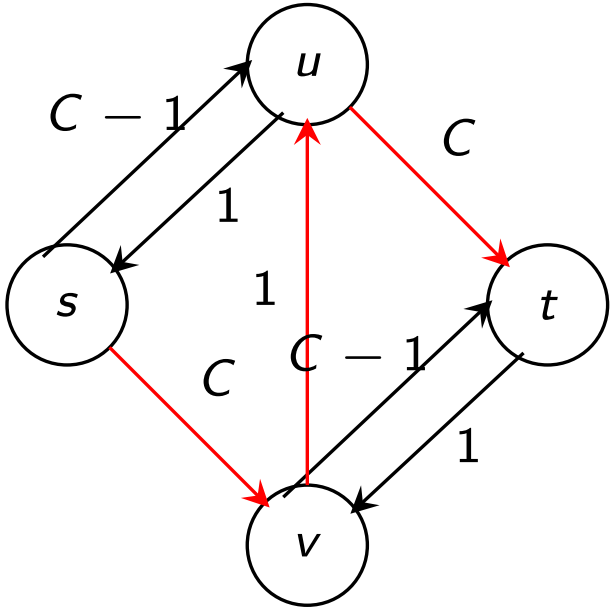

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Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way? Yes! Two variants.

- Choose the augmenting path with largest bottleneck capacity.
- Choose the shortest augmenting path.

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Book gives a simpler variant called Capacity Scaling algorithm that runs in $O(m^2 \log C)$ time.

Removing Dependence on C

- [Edmonds-Karp, Dinitz] Picking augmenting paths with fewest number of edges yields a $O(m^2 n)$ algorithm, i.e., independent of C. Such an algorithm is called a strongly polynomial time algorithm since the running time does not depend on the numbers (assuming RAM model). (Many implementation of Ford-Fulkerson would actually use shortest augmenting path if they use BFS to find an s-t path).

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- Further improvements can yield algorithms running in $O(mn \log n)$, or $O(n^3)$

Finding a Minimum Cut

Question: How do we find an actual minimum s-t cut?

Finding a Minimum Cut

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Proof gives the algorithm!

- Compute an s-t maximum flow f in G
- Obtain the residual graph G_f
- Find the nodes A reachable from s in G_f
- Output the cut $(A, B) = \{(u, v) \mid u \in A, v \in B\}$. Note: The cut is found in G while A is found in G_f

Running time is essentially the same as finding a maximum flow.

Note: Given G and a flow f there is a linear time algorithm to check if f is a maximum flow and if it is, outputs a minimum cut. How?