CS 473: Algorithms

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Part I

All Pairs Shortest Paths

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.
- Find shortest paths for all pairs of nodes.

Single-Source Shortest Paths

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Dijkstra's algorithm for non-negative edge lengths. Running time: $O((m + n) \log n)$ with heaps and $O(m + n \log n)$ with advanced priority queues.

Bellman-Ford algorithm for arbitrary edge lengths. Running time: $O(nm)$.

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Apply single-source algorithms n times, once for each vertex.

- Non-negative lengths. $O(nm \log n)$ with heaps and $O(nm + n^2 \log n)$ using advanced priority queues.
- Arbitrary edge lengths: $O(n^2 m)$.

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- Non-negative lengths. $O(nm \log n)$ with heaps and $O(nm + n^2 \log n)$ using advanced priority queues.
- Arbitrary edge lengths: $O(n^2 m)$. Can we do better?

Shortest Paths and Recursion

- Can we compute the shortest path distance from s to t recursively?
- What are the smaller sub-problems?

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Lemma

Let G be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

- *$s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from s to v_i*

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Sub-problem idea: paths of fewer hops/edges

Hop-based Recursion for Single-Source Shortest Paths

Single-source problem: fix source s.

$OPT(v, k)$: shortest path dist from s to v using at most k edges.

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Recursion for $OPT(v, k)$:

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Recursion for $OPT(v,k)$:

$$OPT(v,k) = \min_{u \in V}(OPT(u,k-1) + c(u,v)).$$

Base case: $OPT(v,1) = c(s,v)$ if $(s,v) \in E$ otherwise ∞

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Leads to Bellman-Ford algorithm — see text book.

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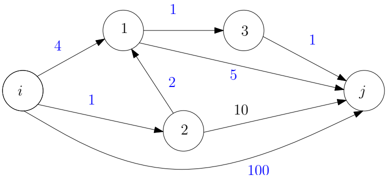
$OPT(v, k)$ values are also of independent interest: shortest paths with at most k hops

All-Pairs: recursion on the index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $dist(i, j, k)$: shortest path distance between v_i and v_j among all paths in which the largest index of an *intermediate node* is at most k



$$
\begin{aligned}
dist(i, j, 0) &= \\
dist(i, j, 1) &= \\
dist(i, j, 2) &= \\
dist(i, j, 3) &=
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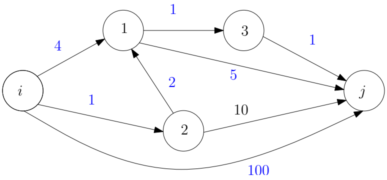

$$
\begin{aligned}
dist(i, j, 0) &= 100 \\
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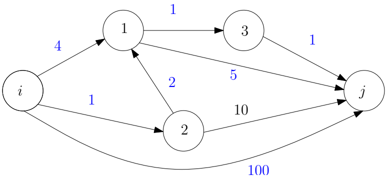

$$\begin{aligned} dist(i,j,0) &= 100 \\ dist(i,j,1) &= 9 \\ dist(i,j,2) &= \\ dist(i,j,3) &= \end{aligned}$$

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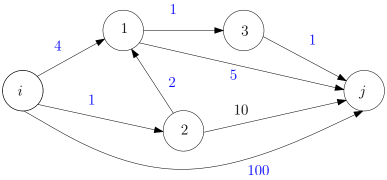

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\begin{aligned}
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dist(i, j, 1) &= 9 \\
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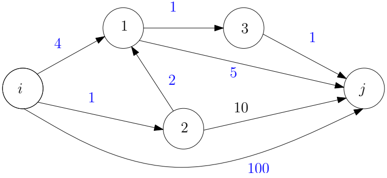

$$
\begin{aligned}
dist(i, j, 0) &= 100 \\
dist(i, j, 1) &= 9 \\
dist(i, j, 2) &= 8 \\
dist(i, j, 3) &= 5
\end{aligned}
$$

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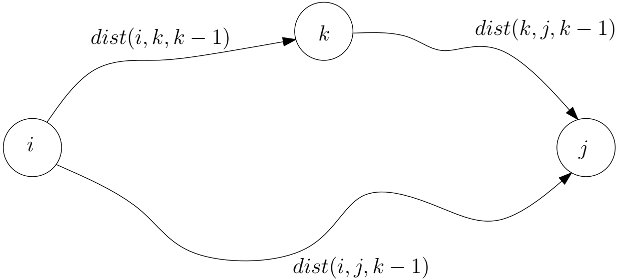
What is $dist(i,j)$?

~~d~~ $dist(i,j) = dist(i,j,n)$

All-Pairs: recursion on the index of intermediate nodes



$$dist(i,j,k) = \min(dist(i,j,k-1), dist(i,k,k-1) + dist(k,j,k-1))$$

Base case: $dist(i,j,0) = c(i,j)$ if $(i,j) \in E$, otherwise ∞

Correctness: If $i \to j$ shortest path goes through k then k occurs only once on the path — otherwise there is a negative length cycle.

Floyd-Warshall Algorithm for All-Pairs Shortest Paths

```
Check if G has a negative cycle using Bellman-Ford in O(mn) time
If there is a negative cycle return

for i=1 to n do
   for j=1 to n do
    dist(i,j,0) = c(i,j)

for k = 1 to n do
   for i=1 to n do
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Correctness: Recursion works under the assumption that all shortest paths are defined (no negative length cycle).
Running Time:

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Running Time: $\Theta(n^3)$

Part II

Knapsack

Knapsack Problem

Input Given a Knapsack of capacity W lbs. and n objects with ith object having weight w_i and value v_i; assume W, w_i, v_i are all positive integers

Goal Fill the Knapsack without exceeding weight limit while maximizing value.

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Basic problem that arises in many applications as a sub-problem.

Knapsack Example

Example

Item	1	2	3	4	5
Value	1	6	18	22	28
Weight	1	2	5	6	7

If $W = 11$, the best is $\{3, 4\}$ giving value 40.

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Special Case

When $v_i = w_i$, the Knapsack problem is called the Subset Sum Problem.

Greedy Approach

- Pick objects with greatest value

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  - Let $W = 2$, $w_1 = w_2 = 1$, $w_3 = 2$, $v_1 = v_2 = 2$ and $v_3 = 3$; greedy strategy will pick $\{3\}$, but the optimal is $\{1, 2\}$

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  - Let $W = 4$, $w_1 = w_2 = 2$, $w_3 = 3$, $v_1 = v_2 = 3$ and $v_3 = 5$; greedy strategy will pick $\{3\}$, but the optimal is $\{1, 2\}$

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  - Let $W = 4$, $w_1 = w_2 = 2$, $w_3 = 3$, $v_1 = v_2 = 3$ and $v_3 = 5$; greedy strategy will pick $\{3\}$, but the optimal is $\{1, 2\}$
  - Can show that a slight modification always gives half the optimum profit: pick the better of the output of this algorithm and the largest value item. Also, the algorithms gives better approximations when all item weights are small when compared to W.

Towards a Recursive Solution

First guess: $\mathrm{Opt}(i)$ is the optimum solution value for items $1, \ldots, i$.

Observation

Consider an optimal solution $\mathcal{O}$ for $1, \ldots, i$

Case item $i \notin \mathcal{O}$ *$\mathcal{O}$ is an optimal solution to items 1 to $i - 1$*

Case item $i \in \mathcal{O}$ *Then $\mathcal{O} - \{i\}$ is an optimum solution for items 1 to $n - 1$ in knapsack of capacity $W - w_i$.*

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Subproblems depend also on remaining *capacity.*
Cannot write subproblem only in terms of $\mathrm{Opt}(1), \ldots, \mathrm{Opt}(i - 1)$.

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$\mathrm{Opt}(i, w)$: optimum profit for items 1 to i in knapsack of size w

Goal: compute $\mathrm{Opt}(n, W)$

Dynamic Programming Solution

Definition

Let $\mathrm{Opt}(i, w)$ be the optimal way of picking items from 1 to i, with total weight not exceeding w

$$\mathrm{Opt}(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ \mathrm{Opt}(i-1, w) & \text{if } w_i > w \\ \max\{\mathrm{Opt}(i-1, w), \mathrm{Opt}(i-1, w-w_i) + v_i\} & \text{otherwise} \end{cases}$$

An Iterative Algorithm

```
for w = 0 to W
    M[0,w] = 0
for i = 1 to n
    for w = 1 to W
        if (w_i > w)
            M[i,w] = M[i-1,w]
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Running Time

- Time taken is $O(nW)$
- Input has size $O(n + \log W)$; so running time not polynomial but "pseudo-polynomial"!

Knapsack Algorithm and Polynomial time

Input size for Knapsack:

Knapsack Algorithm and Polynomial time

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Knapsack Algorithm and Polynomial time

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Running time of dynamic programming algorithm: $O(nW)$

Not a polynomial time algorithm.
Example: $W = 2^n$ and $w_i, v_i \in [1..2^n]$.
Input size is $O(n^2)$, running time is $O(n2^n)$ arithmetic/comparisons.

Algorithm is called a *pseudo-polynomial* time algorithm because running time is polynomial if *numbers* in input are of size polynomial in *combinatorial* size of problem.

Knapsack is NP-hard if numbers are not polynomial in n.

Eg $\quad v_1 = 10,\ w_1 = 3000$

$v_2 = 5,\ w_2 = 100000$

$\vdots$

$v_n = 10,\ w_n = 50{,}000$

$$W = 200{,}000000$$

$\forall \varepsilon > 0$

$(1-\varepsilon)$ approx in time $\frac{1}{\varepsilon} n \log n$

Part III

Traveling Salesman Problem

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Input A graph $G = (V, E)$ with non-negative edge costs/lengths. $c(e)$ for edge e

Goal Find a tour of minimum cost that visits each node.

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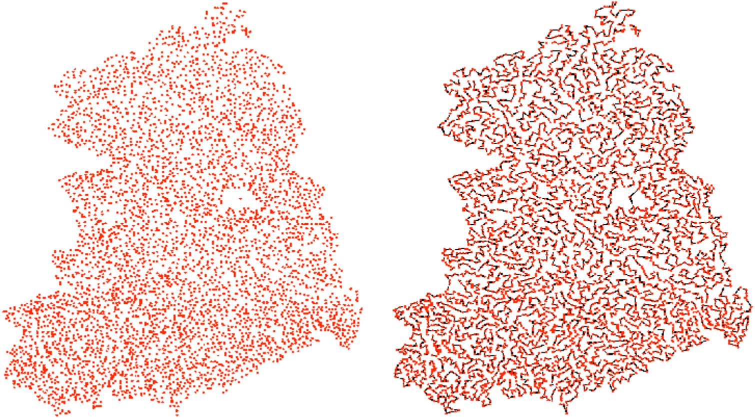
No polynomial time algorithm known. Problem is NP-Hard.

Example: optimal tour for cities of a country (which one?)



An Exponential Time Algorithm

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Can we do better? Can we get a $2^{O(n)}$ time algorithm?

Towards a Recursive Solution

- Order vertices as $v_1, v_2, \ldots, v_n$
- $OPT(S)$: optimum TSP tour for the vertices $S \subseteq V$ in the graph restricted to S. Want $OPT(V)$.

Can we compute $OPT(S)$ recursively?

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- Say $v \in S$. What are the two neighbors of v in optimum tour in S?
- If u, w are neighbors of v in an optimum tour of S then removing v gives an optimum *path* from u to w visiting all nodes in $S - \{v\}$.

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Path from u to w is not a recursive subproblem! Need to find a more general problem to allow recursion.

A More General Problem: TSP Path

Input A graph $G = (V, E)$ with non-negative edge costs/lengths($c(e)$ for edge e) and two nodes s, t

Goal Find a path from s to t of minimum cost that visits each node exactly once.

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Can solve TSP using above. Do you see how?

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What is the next node in the optimum path from u to v? Suppose it is w. Then what is $OPT(u, v, S)$?

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What is the next node in the optimum path from u to v? Suppose it is w. Then what is $OPT(u, v, S)$?

$$OPT(u, v, S) = c(u, w) + OPT(w, v, S - \{u\})$$

We do not know w! So try all possibilities for w.

A Recursive Solution

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Disadvantage of dynamic programming solution:

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Disadvantage of dynamic programming solution: memory!

Dynamic Programming: Postscript

Dynamic Programming = Smart Recursion + Memoization

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- How to come up with the recursion?
- How to recognize that dynamic programming may apply?

Some Tips

- Problems where there is a *natural* linear ordering: sequences, paths, intervals, DAGs etc. Recursion based on ordering (left to right or right to left or topological sort) usually works.
- Problems involving trees: recursion based on subtrees.
- More generally:
  - Problem admits a natural recursive divide and conquer
  - If optimal solution for whole problem can be simply composed from optimal solution for each separate pieces then plain divide and conquer works directly
  - If optimal solution depends on all pieces then can apply dynamic programming if *interface/interaction* between pieces is *limited*. Augument recursion to not simply find an optimum solution but also an optimum soluton for each possible way to interact with the other pieces.

Examples

- Longest Increasing Subsequence: break sequence in the middle say. What is the interaction between the two pieces in a solution?
- Sequence Alignment: break both sequences in two pieces each. What is the interaction between the two sets of pieces?
- Independent Set in a Tree: break tree at root into subtrees. What is the interaction between the sutrees?
- Independent Set in an graph: break graph into two graphs. What is the interaction? Very high!
- Knapsack: Split items into two sets of half each. What is the interaction?