CS 473: Algorithms

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Part I

Introduction to Dynamic Programming

Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction

- reduce problem to a *smaller* instance of *itself*
- self-reduction

Recursion

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Recursion: a special case of reduction

- reduce problem to a *smaller* instance of *itself*
- self-reduction

- Problem instance of size n is reduced to one or more instances of size $n-1$ or less.
- For termination, problem instances of small size are solved by some other method as *base cases*

Recursion in Algorithm Design

- Tail Recursion: problem reduced to a *single* recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms. Examples: Interval scheduling, MST algorithms, etc.
- Divide and Conquer: problem reduced to multiple *independent* sub-problems that are solved separately. Conquer step puts together solution for bigger problem.
- Dynamic Programming: problem reduced to multiple (typically) *dependent or overlapping* sub-problems. Use *memoization* to avoid recomputation of common solutions leading to *iterative bottom-up* algorithm.

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting and amazing properties.
A journal *The Fibonacci Quarterly*!

- $F(n) = (\phi^n - (1-\phi)^n)/\sqrt{5}$ where ϕ is the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$.
- $\lim_{n\to\infty} F(n+1)/F(n) = \phi$

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- $\lim_{n\to\infty} F(n+1)/F(n) = \phi$

Question: Given n, compute $F(n)$.

Recursive Algorithm for Fibonacci Numbers

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n-1) + Fib(n-2)
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Roughly same as $F(n)$

$$T(n) = \Theta(\phi^n)$$

Thus algorithm does exponential in n additions.

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Thus algorithm does exponential in n additions. Can we do better?

An iterative algorithm for Fibonacci numbers

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Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        F[0] = 0
        F[1] = 1
        for i = 2 to n do
            F[i] = F[i-1] + F[i-2]
        return F[n]
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What is the running time of the algorithm? $O(n)$ additions.

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- Iterative algorithm is storing computed values and building bottom up the final value.

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Dynamic Programming: finding a recursion that can be *effectively/efficiently* memoized

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.

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    else if (Fib(n) was previously computed)
        return stored value of Fib(n)
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```

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)

Automatic explicit memoization

Initialize table/array M of size n such that $M[i] = -1$ for $0 \leq i < n$

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Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else if (M[n] ≠ -1) (* M[n] has stored value of Fib(n) *)
        return M[n]
    else
        M[n] = Fib(n-1) + Fib(n-2)
        return M[n]
```

Need to know upfront the number of subproblems to allocate memory

Automatic implicit memoization

Initialize a (dynamic) dictionary data structure D to empty

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else if (n is already in D)
        return value stored with n in D
    else
        val = Fib(n-1) + Fib(n-2)
        Store (n, val) in D
        return val
```

Explicit vs Implicit Memoization

- Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.
- Implicit and automatic memoization used when problem structure or algorithm is either not well understood or in fact unknown to the underlying system
  - need to pay overhead of datastructure
  - Functional languages such as LISP automatically do memoization, usually via hashing based dictionaries.

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- Hence output size is exponential in input size so no polynomial time algorithm possible!
- Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?
- Running time of recursive algorithm is $O(n\phi^n)$ but can in fact shown to be $O(\phi^n)$ by being careful. Doubly exponential in input size and exponential even in output size.

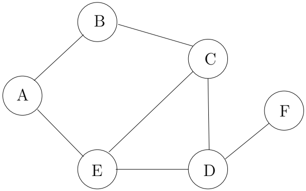
Part II

Recursion and Brute Force Search

Maximum Independent Set in a Graph

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in S. That is, if $u, v \in S$ then $(u, v) \notin E$.

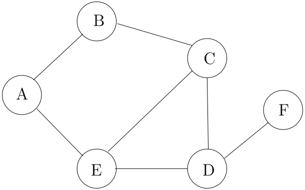


Some independent sets in graph above:

Maximum Independent Set Problem

Input Graph $G = (V, E)$

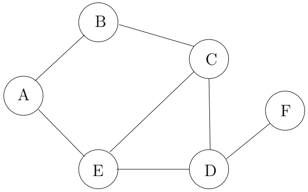
Goal Find maximum sized independent set in G



Maximum Weight Independent Set Problem

Input Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal Find maximum weight independent set in G



Maximum Weight Independent Set Problem

- No one knows an *efficient* (polynomial time) algorithm for this problem
- Problem is NP-Complete and it is *believed* that there is no polynomial time algorithm

A *brute-force* algorithm: try all subsets of vertices.

Brute-force enumeration

Algorithm to find the size of the maximum weight independent set.

```
MaxIndSet(G = (V, E)):
    max = 0
    for each subset S ⊆ V
        check if S is an independent set
        if S is an independent set and w(S) > max
            max = w(S)
        endfor
    Output max
```

Brute-force enumeration

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```

Running time: suppose G has n vertices and m edges

- 2^n subsets of V
- checking each subset S takes $O(m)$ time
- total time is $O(m2^n)$

A Recursive Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$.
For a vertex u let $N(u)$ be its neighbours.

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Observation

One of the following two cases is true

- Case 1: v_n *is in* some *maximum independent set.*
- Case 2: v_n *is in* no *maximum independent set.*

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Observation

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- Case 1 v_n *is in* some *maximum independent set.*
- Case 2 v_n *is in* no *maximum independent set.*

Recursive-MIS(G):
 If G is empty, Output 0
 $a =$ Recursive-MIS($G - v_n$)
 $b = w(v_n) +$ Recursive-MIS($G - v_n - N(v_n)$)
 Output $\max(a, b)$

Recursive Algorithms of MIS

Running time:

$$T(n) =$$

Recursive Algorithms of MIS

Running time:

$$T(n) = T(n-1) + T(n-1-deg(v_1)) + O(1)$$

where $deg(v_1)$ is the degree of v_1. $T(0) = T(1) = 1$ is base case.

Worst case is when $dev(v_1) = 0$ when the recurrence becomes

$$T(n) = 2T(n-1) + O(1)$$

Solution to this is $T(n) = O(2^n)$.

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Solution to this is $T(n) = O(2^n)$.
Can we improve this?
Worst case is when $deg(v_n) = 0$. In this case v_n is in every maximum weight independent set! No need to check!

An Improved Algorithm

`Recursive-MIS`(G):
  `If` G `is empty, Output 0`
  $a =$ `Recursive-MIS`$(G - v_n)$
  `If` $(deg(v_n) = 0)$
    `Output` $w(v_n) + a$
  `Else`
    $b = w(v_n) +$ `Recursive-MIS`$(G - v_n - N(v_n))$
    `Output` $\max(a, b)$

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Recursive-MIS(G):
    If G is empty, Output 0
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    Else
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        Output max(a, b)
```

Running time:

$$T(n) = \max\{T(n-1), T(n-1) + T(n-2)\} + O(1)$$

Similar to the Fibonacci recurrence. Can show that $T(n) = O(1.618^n)$.

Recursion for MIS: Observations

- We expressed the optimum solution *value* on G recursively as a function of the values on two smaller instances.

$$OPT(G) = \max\{OPT(G - v_n), w(v_n) + OPT(G - v_n - N(v_n))\}$$

- Can we memoize the recursive algorithm(s)? Yes.
- Does memoization improve the running time in the worst case? No. Number of sub-problems can be large (can create explicit graphs).

Part III

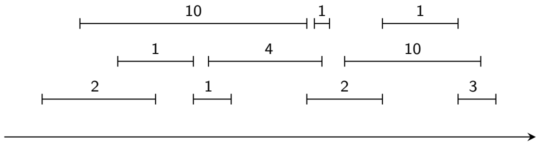
Weighted Interval Scheduling

Weighted Interval Scheduling

Input A set of jobs with start times, finish times and *weights* (or profits)

Goal Schedule jobs so that total weight of jobs is maximized

- Two jobs with overlapping intervals cannot both be scheduled!



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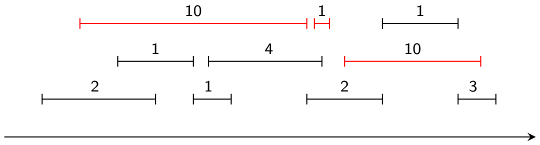
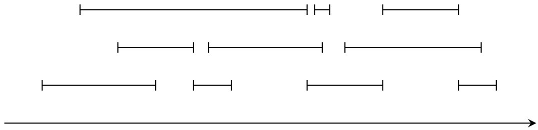


Interval Scheduling

Input A set of jobs with start and finish times to be scheduled on a resource; special case where all jobs have weight 1

Goal Schedule as many jobs as possible



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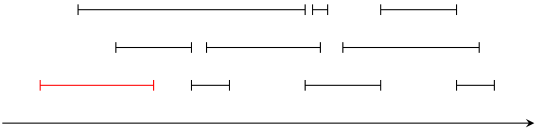

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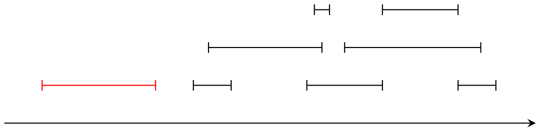

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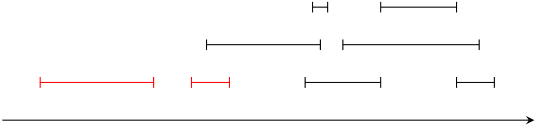

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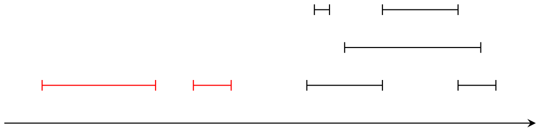

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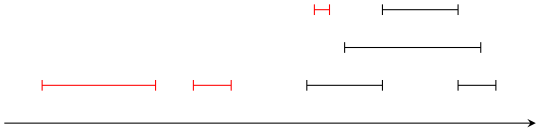

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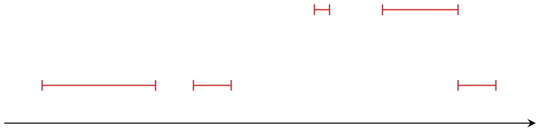

Greedy Strategy for Weighted Interval Scheduling

- Pick jobs in order of finishing times
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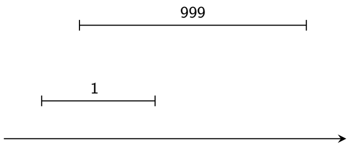

Other Greedy Strategies

- Largest weight/profit first
- Largest weight to length ratio first
- Shortest length first
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None of the above strateies lead to an optimum solution.

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Moral: Greedy strategies often don't work!

Reduction to Max Weight Independent Set Problem

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- Given weighted interval scheduling instance I create an instance of max weight independent set on a graph $G(I)$ as follows.
  - For each interval i create a vertex v_i with weight w_i.
  - Add an edge between v_i and v_j if i and j overlap.
- Claim: max weight independent set in $G(I)$ has weight equal to max weight set of intervals in I that do not overlap

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We do not know an efficient (polynomial time) algorithm for independent set! Can we take advantage of the interval structure to find an efficient algorithm?

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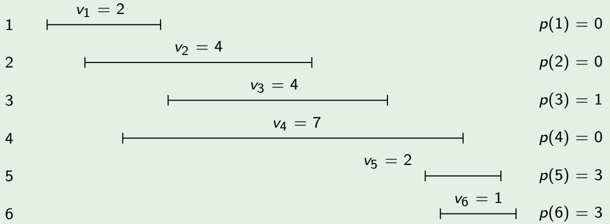
Example



Towards a Recursive Solution

Observation

Consider an optimal schedule $\mathcal{O}$

Case $n \in \mathcal{O}$ *None of the jobs between n and $p(n)$ can be scheduled. Moreover $\mathcal{O}$ must contain an optimal schedule for the first $p(n)$ jobs.*

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Case $n \notin \mathcal{O}$ *$\mathcal{O}$ is an optimal schedule for the first $n-1$ jobs!*

A Recursive Algorithm

Notation: O_i value of an optimal schedule for the first i jobs.

Recursively compute $O_{p(n)}$
Recursively compute O_{n-1}
If $(O_{p(n)} + v_n < O_{n-1})$ then
    $O_n = O_{n-1}$
else
    $O_n = O_{p(n)} + v_n$
Output O_n

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Time Analysis

Running time is $T(n) = T(p(n)) + T(n-1) + O(1)$ which is ...

Bad Example

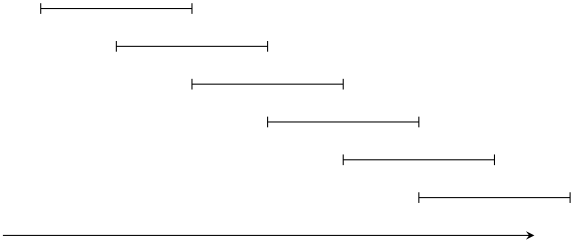

Figure: Bad instance for recursive algorithm

Running time on this instance is

$$T(n) = T(n-1) + T(n-2) + O(1)$$

Bad Example

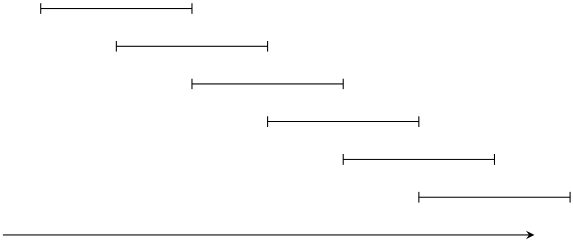

Figure: Bad instance for recursive algorithm

Running time on this instance is

$$T(n) = T(n-1) + T(n-2) + O(1) = \Theta(\phi^n)$$

where $\phi \approx 1.618$ is the golden ratio.

Analysis of the Problem

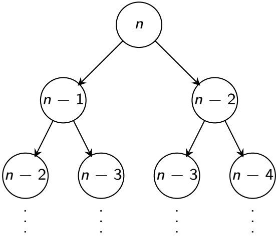

Figure: Label of node indicates size of sub-problem. Tree of sub-problems grows very quickly

Memo(r)ization

Observation

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Memo(r)ization

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- *Number of different sub-problems in recursive algorithm is* $O(n)$*; they are* $O_1, O_2, \ldots, O_{n-1}$
- *Exponential time is due to recomputation of solutions to sub-problems*

Solution

Store optimal solution to different sub-problems, and perform recursive call only if not already computed.

Recursive Solution with Memoization

```
computeOpt(int j)
    if j = 0 then return 0
    if M[j] is defined then (* sub-problem already solved *)
        return M[j]
    if M[j] is not defined then
        M[j] = max(v_j + computeOpt(p(j)), computeOpt(j-1))
        return M[j]
```

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Time Analysis

- Each invocation, $O(1)$ time plus: either return a computed value, or generate 2 recursive calls and fill one M[·]

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Time Analysis

- Each invocation, $O(1)$ time plus: either return a computed value, or generate 2 recursive calls and fill one M[·]
- Initially no entry of M[] is filled; at the end all entries of M[] are filled
- So total time is $O(n)$

Automatic Memoization

Fact

Many functional languages (like LISP) automatically do memoization for recursive function calls!

Back to Weighted Interval Scheduling

Iterative Solution

```
M[0] = 0
for i = 1 to n
    M[i] = max(v_i + M[p(i)], M[i-1])
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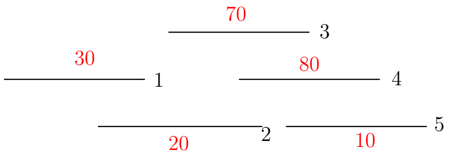
M: table of subproblems

- There is always a table in dynamic programming
- Recursion determines order in which table is filled up
- Think of decomposing problem first (recursion) and then worry about setting up table — this comes naturally from recursion

Example



$$p(5) = 2,\ p(4) = 1,\ p(3) = 1,\ p(2) = 0,\ p(1) = 0$$

Computing Solutions

- Memoization + Recursion/Iteration allows one to compute the optimal value. What about the actual schedule?

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```
M[0] = 0
S[0] is empty schedule
for i = 1 to n
    M[i] = max(v_i + M[p(i)], M[i-1])
    S[i] = v_i + M[p(i)] < M[i-1] ?  S[i-1] :  S[p(i)] ∪ {i}
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- Naïvely updating `S[]` takes $O(n)$ time

Computing Solutions: First Attempt

- Memoization + Recursion/Iteration allows one to compute the optimal value. What about the actual schedule?

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for i = 1 to n
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```

- Naïvely updating S[] takes $O(n)$ time
- Total running time is $O(n^2)$

Computing Implicit Solutions

Observation

Solution can be obtained from `M[]` *in $O(n)$ time, without any additional information*

```
findSolution(int j)
    if (j=0) then return empty schedule
    if (v_j + M[p(j)] > M[j-1]) then
        return findSolution(p(j)) ∪ {j}
    else
        return findSolution(j-1)
```

Makes $O(n)$ recursive calls, so `findSolution` *runs in $O(n)$ time.*

Computing Implicit Solutions

A generic strategy for computing solutions in dynamic programming:

- keep track of the *decision* in computing the optimum value of a sub-problem. decision space depends on recursion
- once the optimum values are computed, go back and use the decision values to compute an optimum solution.

Question: What is the decision in computing $M[i]$?

Computing Implicit Solutions

A generic strategy for computing solutions in dynamic programming:

- keep track of the *decision* in computing the optimum value of a sub-problem. decision space depends on recursion
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Question: What is the decision in computing $M[i]$? Whether to include i or not.

Computing Implicit Solutions

```
M[0] = 0
for i = 1 to n
    M[i] = max(v_i + M[p(i)], M[i-1])
    if (v_i + M[p(i)] > M[i-1])
        Decision[i] = 1 (* 1 means i included in solution M[i] *)
    else
        Decision[i] = 0 (* 0 means i not included in solution M[i] *)

S = ∅, i = n
While (i > 0) do
    if (Decision[i] == 1)
        S = S ∪ i
        i = p(i)
    else
        i = i-1

Output S
```

Part IV

Longest Increasing Subsequence

Sequences

Definition

Sequence: an ordered list $a_1, a_2, \ldots, a_n$. *Length* of a sequence is number of elements in the list.

Definition

$a_{i_1}, \ldots, a_{i_k}$ is a subsequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < \ldots < i_k \leq n$.

Definition

A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Subsequence: 5, 2, 1

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an *increasing subsequence* $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

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Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

A First Recursive Approach

$L(i)$: length of longest increasing subsequence in $a_1, a_2, \ldots, a_i$.

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Case 1 : $L(i)$ does not contain a_i, then $L(i) = L(i-1)$
Case 2 : $L(i)$ contains a_i, then $L(i) =?$

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Case 2 : $L(i)$ contains a_i, then $L(i) =?$ What is the element in the subsequence before a_i? If it is a_j then it better be the case that $a_j < a_i$ since we are looking for an increasing sequence.

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$$L(i) = 1 + \max_{j<i \text{ and } a_j<a_i} L(j)$$

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$$L(i) = 1 + \max_{j<i \text{ and } a_j<a_i} L(j)$$

Is the above correct? No, because we do not know that $L(j)$ corresponds to a subsequence that actually ends at a_j!

A Correct Recursion

$L(i)$: longest increasing subsequence in $a_1, a_2, \ldots, a_i$ that ends in a_i

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How many subproblems? $O(n)$

Running time for Recursion

$$L(i) = 1 + \max_{j<i \text{ and } a_j<a_i} L(j)$$

$T(i)$: time to compute $L[i]$:

$$T(i) = 1 + \sum_{j=1}^{i-1} T(i-1) \quad \text{and} \quad T(1) = 1.$$

$T(n) =$

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$$T(i) = 1 + \sum_{j=1}^{i-1} T(i-1) \quad \text{and} \quad T(1) = 1.$$

$T(n) = 2^{n-1}$.

Iterative Algorithm via Memoization

Recurrence:

$$L(i) = 1 + \max_{j<i \text{ and } a_j<a_i} L(j)$$

Iterative algorithm:

```
for i = 1 to n do
    L[i] = 1
    for j = 1 to i-1 do
        if (a_j < a_i) and (1+L[j]) > L[i] then
            L[i] = 1 + L[j]
```

Output $\max_{i=1}^{n}$ L[i]

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Output $\max_{i=1}^{n}$ L[i]

Running Time: $O(n^2)$
Space: $O(n)$

Computing an Optimum Solution

Keep track of decision when computing $L[i]$.

```
for i = 1 to n do
    L[i] = 1
    prev[i] = 0 (* 0 is a sentinel value *)
    for j = 1 to i-1 do
        if (a_j < a_i) and (1+L[j]) > L[i] then
            L[i] = 1 + L[j]
            prev[i] = j
```

Output $\max_{i=1}^{n}$ L[i]

Exercise: show how to output an increasing sequence of length equal to $L[i]$ using the prev pointers.