Problem 1. An assignment for a computer graphics course requires students to write a program that rotates pictures by 90 degrees. Assume that the picture is square and has size $2^n \times 2^n$ pixels. Give a divide-and-conquer algorithm that makes 4 recursive calls, and 5 block transfers per call. A block transfer is a library routine that copies a square block of pixels from one location to another; this can be implemented fairly quickly. How many block transfers does your algorithm perform in total?

Problem 2 ([3.10]). Let $G = (V, E)$ be an unweighted, undirected graph and let $u$ and $v$ be two vertices of $G$. Describe a linear time algorithm to find the number of shortest paths from $u$ to $v$. Note that we only want the number of paths as there may be an exponential number of them. Give an example graph with an exponential number of $(u, v)$-paths.

Problem 3 ([6.6]). You are tasked with a writing a “pretty print” algorithm for a word processor. You are given a sequence of words, $W = \{w_1, \ldots, w_n\}$ where word $w_i$ consists of $c_i$ characters. You are also given a maximum line length $L$. A formatting of $W$ consists of a partition of the words in $W$ into lines. In the words assigned to a single line, there should be a space after each word except the last; and so if $w_j, w_{j+1}, \ldots, w_k$ are assigned to one line, then we should have

$$\sum_{i=j}^{k-1} (c_i + 1) + c_k \leq L.$$

We will call an assignment of words to a line valid if it satisfies this inequality. The difference between the left-hand side and the right-hand side will be called the slack of the line—that is, the number of spaces left at the right margin.

Give an efficient algorithm to find a partition of a set of words $W$ into valid lines, so that the sum of the squares of the slacks of all lines (including the last line) is minimized.
Problem 4. You are given a flow network $G$ with source $s$ and sink $t$. We call $(S, V \setminus S)$ an $(s, t)$-cut if $s \in S$ and $t \in T$. Define the intersection of two cuts $(S, V \setminus S)$ and $(S', V \setminus S')$ to be $(S \cap S', V \setminus (S \cap S'))$. Union is defined similarly. Show that the intersection of two minimum cuts is a minimum cut and that the union of two minimum cuts is a minimum cut.

Problem 5. Answer both questions on NP-completeness below.

- ([8.17]) You are given a directed graph $G = (V, E)$ with weights $w_e$ on each edge $e \in E$. The weights can be negative or positive. The Zero-Weight-Cycle Problem is to decide if there is a simple cycle in $G$ such that the sum of the edge weights on this cycle is exactly 0. Prove that this problem is NP-Complete.

- A SAT formula $\phi$ is said to be monotone if each clause has only positive literals. For example,

$$
(x_1 \lor x_4 \lor x_5) \land (x_2 \lor x_3 \lor x_4) \land (x_3 \lor x_5)
$$

is a monotone formula. It is easy to see that a monotone formula is always satisfiable — simply set each variable to 1. It is however natural to ask for a satisfying assignment with as few variables as possible set to 1. For example, in the above formula we can set two variables to 1 (say $x_4$ and $x_3$) and satisfy it. Motivated by this we consider the problem Monotone SAT with Few True Variables. The input to this problem is a monotone SAT formula $\phi$ and an integer $k$. The question is whether $\phi$ is satisfiable with at most $k$ variables set to 1 (and the rest to 0). Prove that this problem is NP-Complete. Hint: Consider a reduction from Vertex Cover.

Problem 6. Consider the Partition problem. You are given integers $x_1, \ldots, x_n$, and you want to decide whether the numbers can be partitioned into two subsets $S_1$ and $S_2$ with the same sum:

$$
\sum_{x_i \in S_1} x_i = \sum_{x_j \in S_2} x_j.
$$

- Show that Partition is NP-Complete.

- Let $T$ be the sum of the elements so that $T = \sum_{i=1}^{n} x_i$. Describe an algorithm for Partition that is polynomial in $n$ and $T$.

- Why do the above not prove that $P = NP$?