ODE Stability

Stephen Bond

UIUC

CS 450, Spring 2010
Solve the Ordinary Differential Equation

\[ \frac{dy}{dt} = \lambda y, \quad \text{with} \quad y(0) = y_0 \]

Exact Solution

\[ y(t) = y_0 e^{\lambda t} \]

Solution is \textit{asymptotically stable} or \textit{decaying in magnitude} if \( \text{Re}(\lambda) < 0 \)

Solution is \textit{stable} or \textit{not growing in magnitude} if \( \text{Re}(\lambda) \leq 0 \)
"Stable" values of $\lambda$ in Complex Plane:

$$\lim_{t \to \infty} e^{\lambda t} = ?$$

$\text{Re} \ (\lambda) > 0 \quad (\text{Unstable})$

$\text{Re} \ (\lambda) \leq 0 \quad (\text{Stable})$

Left Half-Plane = Stable

Right Half-Plane = Unstable
Euler’s Method

- Euler’s Method

\[ y_{k+1} = y_k + hf(t_k, y_k) \]

- Apply Euler’s Method to

\[ \frac{dy}{dt} = \lambda y, \quad \text{with} \quad y(0) = y_0 \]

we get

\[ y_{k+1} = y_k + h\lambda y_k \]

- Regrouping terms

\[ y_{k+1} = (1 + h\lambda)y_k = (1 + h\lambda)^{k+1}y_0 \]

- Euler’s Method is stable if

\[ |1 + h\lambda| \leq 1 \]
"Stable" values of $h\lambda$ in Complex Plane:

$$\lim_{k \to \infty} |1 + h\lambda|^k = ?$$

$|1 + h\lambda| > 1$ (Unstable)

$|1 + h\lambda| \leq 1$ (Stable)

For $\text{Re}(\lambda) < 0$, method is Conditionally Stable

For $\text{Re}(\lambda) > 0$, method is Unconditionally Unstable
Backward Euler’s Method

- Backward Euler’s Method
  \[ y_{k+1} = y_k + hf(t_{k+1}, y_{k+1}) \]

- Apply Backward Euler’s Method to
  \[ \frac{dy}{dt} = \lambda y, \quad \text{with} \quad y(0) = y_0 \]
  
  we get
  \[ y_{k+1} = y_k + h \lambda y_{k+1} \]

- Regrouping terms
  \[ y_{k+1} = (1 - h\lambda)^{-1} y_k = \left( \frac{1}{1 - h\lambda} \right)^{k+1} y_0 \]

- Backward Euler’s Method is stable if
  \[ \frac{1}{|1 - h\lambda|} \leq 1 \]
"Stable" values of $h\lambda$ in Complex Plane:

$$\lim_{k \to \infty} \frac{1}{|1 - h\lambda|^k} = ?$$

$|1 - h\lambda| < 1$ (Unstable)

$|1 - h\lambda| \geq 1$ (Stable)

For $\text{Re}(\lambda) < 0$, method is Unconditionally Stable

For $\text{Re}(\lambda) > 0$, method is Conditionally Stable or Conditionally Unstable
Trapezoid Method

- Trapezoid Method
  \[ y_{k+1} = y_k + \frac{h}{2} f(t_k, y_k) + \frac{h}{2} f(t_{k+1}, y_{k+1}) \]

- Apply Trapezoid Method to
  \[ \frac{dy}{dt} = \lambda y, \quad \text{with} \quad y(0) = y_0 \]
  we get
  \[ y_{k+1} = y_k + \frac{h}{2} \lambda y_k + \frac{h}{2} \lambda y_{k+1} \]

- Regrouping terms
  \[ y_{k+1} = \frac{1 + h\lambda/2}{1 - h\lambda/2} y_k = \left(\frac{1 + h\lambda/2}{1 - h\lambda/2}\right)^{k+1} y_0 \]

- Trapezoid Method is stable if
  \[ \left| \frac{1 + h\lambda/2}{1 - h\lambda/2} \right| \leq 1 \]
"Stable" values of $h\lambda$ in Complex Plane:

$$\lim_{k \to \infty} \left| \frac{1 + h\lambda/2}{1 - h\lambda/2} \right|^k = ?$$

$|1 + h\lambda/2| > |1 - h\lambda/2|$ (Unstable)

$|1 + h\lambda/2| \leq |1 - h\lambda/2|$ (Stable)

For $\text{Re}(\lambda) < 0$, method is Unconditionally Stable

For $\text{Re}(\lambda) > 0$, method is Unconditionally Unstable
# Linear Stability Summary

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Question: What about non-linear problems?

\[ \frac{dy}{dt} = f(t, y) \quad \text{with} \quad y(0) = y_0 \]

Answer: Use the derivative of \( f \):

Let \( \lambda = \frac{\partial f}{\partial y} \) and check stability

for maximum and minimum values of \( \lambda \).
Question: What about linear systems of ODEs?

\[ \frac{d\vec{y}}{dt} = A\vec{y} + g(t) \quad \text{with} \quad \vec{y}(0) = \vec{y}_0 \]

Answer: Use the eigenvalues of $A$:

Let $\lambda_i$ be the eigenvalues of $A$,

and check stability for each $\lambda_i$.

- Stable if all $\lambda_i$ satisfy stability conditions!
- Unstable if any $\lambda_i$ violates stability conditions!
Question: What about nonlinear systems of ODEs?

\[
\frac{d\vec{y}}{dt} = f(t, \vec{y}) \quad \text{with} \quad \vec{y}(0) = \vec{y}_0
\]

Answer: Use the Jacobian of \( f \):

Let \( \lambda_i \) be the eigenvalues of \( J_f := \left[ \frac{\partial f_i}{\partial y_j} \right] \),

and check stability for each eigenvalue \( \lambda_i \).