Monomial Basis Interpolation: Example

- Interpolate 3 Data Points

\[(t_1, y_1) \quad (t_2, y_2) \quad (t_3, y_3)\]

- Monomial Basis

\[p(t) = x_1 + x_2 t + x_3 t^2\]

- Cost to determine coefficients?

\[
\begin{bmatrix}
1 & t_1 & t_1^2 \\
1 & t_2 & t_2^2 \\
1 & t_3 & t_3^2 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\end{bmatrix}
\]

- Cost to evaluate polynomial?

\[p(t) = x_1 + t (x_2 + x_3 t) \quad \text{Horner’s Method}\]
Lagrange Basis Interpolation: Example

- Interpolate 3 Data Points

\[(t_1, y_1) \quad (t_2, y_2) \quad (t_3, y_3)\]

- Lagrange Basis

\[ p(t) = y_1 \frac{(t - t_2)(t - t_3)}{(t_1 - t_2)(t_1 - t_3)} + y_2 \frac{(t - t_1)(t - t_3)}{(t_2 - t_1)(t_2 - t_3)} + y_3 \frac{(t - t_1)(t - t_2)}{(t_3 - t_1)(t_3 - t_2)} \]

- Cost to determine coefficients?

- Cost to evaluate polynomial?
Interpolate 3 Data Points

\[(t_1, y_1) \quad (t_2, y_2) \quad (t_3, y_3)\]

Barycentric Lagrange Basis

\[p(t) = y_1 \frac{(t - t_1)(t - t_2)(t - t_3)}{(t_1 - t_2)(t_1 - t_3)(t - t_1)} + y_2 \frac{(t - t_1)(t - t_2)(t - t_3)}{(t_2 - t_1)(t_2 - t_3)(t - t_2)} + y_3 \frac{(t - t_1)(t - t_2)(t - t_3)}{(t_3 - t_1)(t_3 - t_2)(t - t_3)}\]

Common term

\[\ell(t) = (t - t_1)(t - t_2)(t - t_3)\]

can be factored out!
Standard vs. Barycentric: Example

- **Lagrange Basis**

\[ p(t) = y_1(t - t_2)(t - t_3)w_1 + y_2(t - t_1)(t - t_3)w_2 + y_3(t - t_1)(t - t_2)w_3 \]

- **Weight Constants**

\[ w_1 = \frac{1}{(t_1 - t_2)(t_1 - t_3)}, \quad w_2 = \frac{1}{(t_2 - t_1)(t_2 - t_3)}, \]

\[ w_3 = \frac{1}{(t_3 - t_1)(t_3 - t_2)} \]

- **Barycentric Lagrange Basis**

\[ p(t) = \ell(t) \left( y_1 \frac{w_1}{t - t_1} + y_2 \frac{w_2}{t - t_2} + y_3 \frac{w_3}{t - t_3} \right) \]

\[ \ell(t) = (t - t_1)(t - t_2)(t - t_3) \]
Barycentric Lagrange: Cost

- Cost to determine weight constants, $w_i$:
  \[ (n \text{ weights}) \times (2n - 2 \text{ operations per weight}) = \mathcal{O}[n^2] \text{ operations}, \]
  which is the same complexity as the Newton basis.

- Cost to evaluate polynomial for given weight constants:
  \[
  \ell(t) \text{ costs } \mathcal{O}[n] \quad \text{and} \quad \sum_{i=1}^{n} y_i \frac{w_i}{t - t_i} \text{ costs } \mathcal{O}[n],
  \]
  which is the same complexity as the Newton basis.

- Cost to update when adding an additional point:
  \[
  \mathcal{O}[n] \text{ for } w_1, \ldots, w_n \quad \text{and} \quad \mathcal{O}[n] \text{ for } w_{n+1},
  \]
  which is the same complexity as the Newton basis.
Barycentric Lagrange: Stability

- What happens when $t = t_i$?
  Use an “if statement” and return $y_i$ to avoid $0/0$.

- What happens as $t \to t_i$?
  Careful analysis shows this is not a problem in floating-point.
  (See Henrici 1979 and Higham 2004)

- Barycentric Lagrange is not “point-order” dependent.
  Same representation independent of point ordering.
  This is in contrast to the Newton basis.

- Weights depend only on $t_i$’s not on $y_i$’s.
  Can be precomputed if interpolation points are fixed.

Consider “Barycentric Lagrange” for your interpolation problem.