

# **CS445 Computational Photography**

## **Linear Algebra Review**

By Yuan Shen

## - Goals of this session:

- Know how to use Jupiter notebook
- Know systems of linear equations in matrix form
- know basic linear algebra notations ( vectors and matrices)
- Know matrix properties (norm, inverse, pseudo-inverse, transpose, rank, eigenvalue, and etc.)
- Know how to setup systems of linear equations and solve them
- Know how to get SVD decomposition of a matrix

## - Reading Material for Linear algebra

- <http://cs229.stanford.edu/summer2019/cs229-linalg.pdf>
- YouTube channel (3Blue1Brown) with visualization:
  - [https://www.youtube.com/channel/UCYO\\_jab\\_esuFRV4b17AJtAw](https://www.youtube.com/channel/UCYO_jab_esuFRV4b17AJtAw)
- CS 357 course website:
  - <https://relate.cs.illinois.edu/course/cs357-s19/page/schedule/>

- System of Linear Equations in Matrix Form

$$\begin{array}{rclclcl} 4x_1 & - & 5x_2 & = & -13 \\ -2x_1 & + & 3x_2 & = & 9. \end{array}$$

- System of Linear Equations in Matrix Form

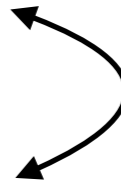
$$\begin{array}{rclcl} 4x_1 & - & 5x_2 & = & -13 \\ -2x_1 & + & 3x_2 & = & 9. \end{array}$$



$$Ax = b$$

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

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## - System of Linear Equations in Matrix Form

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$$Ax = b$$



**Interchanging rows  
has no effect to  
solutions**

$$A = \begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ -13 \end{bmatrix}$$

# Jupyter notebook demo



## - Quiz

- If we know a square matrix  $A \in R^{n \times n}$  has  $n$  linearly independent eigenvectors, then which of the following is/are true:
  - A. The matrix is full ranked
  - B. The matrix is invertible
  - C. The matrix is diagonalizable
  - D. The determinant of  $A$  is not equal to 0
  - E. The number 0 is not an eigenvalue of  $A$

## - Basic Notations:

- The *i*-th element of a vector  $x$  is denoted  $x_i$
- $A \in R^{m \times n}$ ,  $m$  rows and  $n$  columns

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

**Vector**

## - Vector Norm

- A norm of a vector  $\|x\|$  is informally a measure of the “length” of the vector.
- In particular, l2-norm or Euclidean norm is as follows:

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}.$$

## - Vector-vector inner product

For inner product,  $x$  and  $y$  should have the same dimension

Given two vectors  $x, y \in \mathbb{R}^n$ , the quantity  $x^T y$ , sometimes called the *inner product* or *dot product* of the vectors, is a real number given by

$$x^T y \in \mathbb{R} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^n x_i y_i.$$

## - Vector-vector outer-product

For outer product,  $x$  and  $y$  do not have to be in the same dimension

Given vectors  $x \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^n$  (not necessarily of the same size),  $xy^T \in \mathbb{R}^{m \times n}$  is called the *outer product* of the vectors. It is a matrix whose entries are given by  $(xy^T)_{ij} = x_i y_j$ , i.e.,

$$xy^T \in \mathbb{R}^{m \times n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_m y_1 & x_m y_2 & \cdots & x_m y_n \end{bmatrix}.$$

**Matrix**

# - Matrix Multiplication

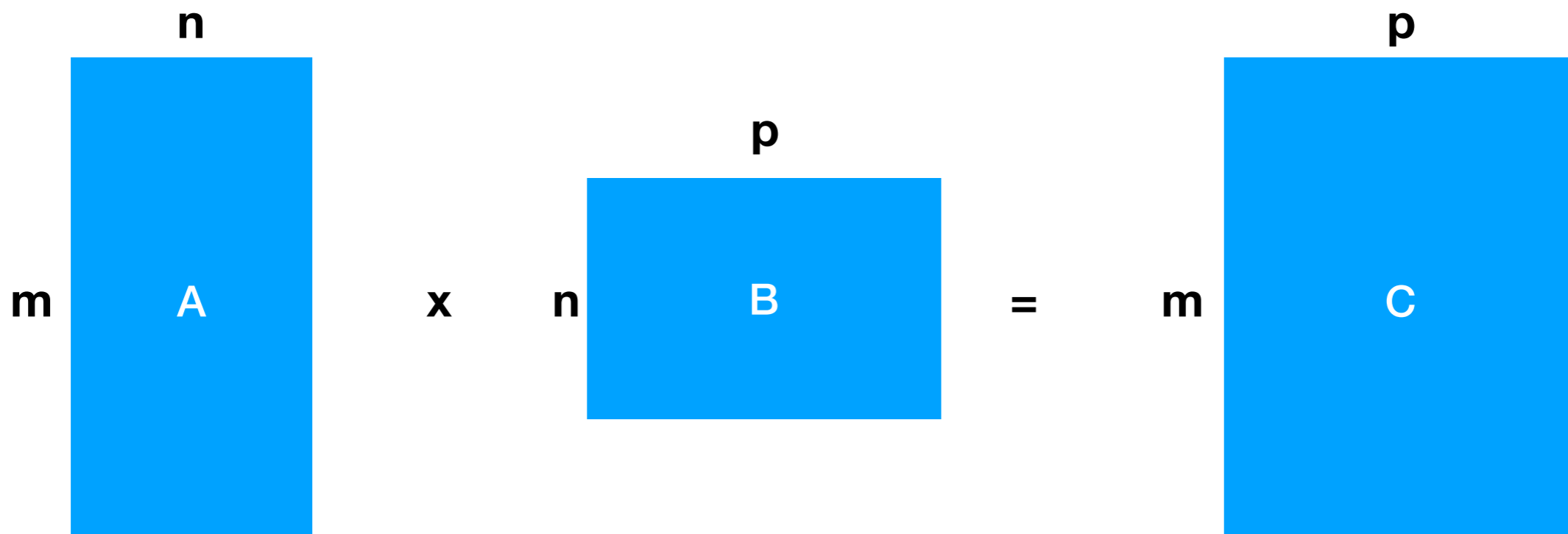
Dimension must match!

The product of two matrices  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times p}$  is the matrix

$$C = AB \in \mathbb{R}^{m \times p},$$

where

$$C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}.$$





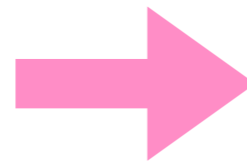
## - Matrix Properties

- Matrix multiplication is associative:  $(AB)C = A(BC)$ .
- Matrix multiplication is distributive:  $A(B + C) = AB + AC$ .
- Matrix multiplication is, in general, *not* commutative; that is, it can be the case that  $AB \neq BA$ . (For example, if  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times q}$ , the matrix product  $BA$  does not even exist if  $m$  and  $q$  are not equal!)

## - Matrix Transpose

**A**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

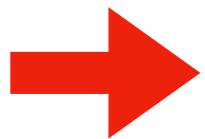


**A<sup>T</sup>**

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

The following properties of transposes are easily verified:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A + B)^T = A^T + B^T$



## - Matrix Rank

- Definition: the number of linearly independent columns/rows of  $A$
- For  $A \in \mathbb{R}^{m \times n}$ ,  $\text{rank}(A) \leq \min(m, n)$ . If  $\text{rank}(A) = \min(m, n)$ , then  $A$  is said to *full rank*.

## - Matrix Inverse

The *inverse* of a square matrix  $A \in \mathbb{R}^{n \times n}$  is denoted  $A^{-1}$ , and is the unique matrix such that

$$A^{-1}A = I = AA^{-1}.$$

Note that not all matrices have inverses. Non-square matrices, for example, do not have inverses by definition.

## - Matrix eigenvalue and eigenvector

We usually normalized the norm of eigenvector so that,  $\|v\|_2 = 1$

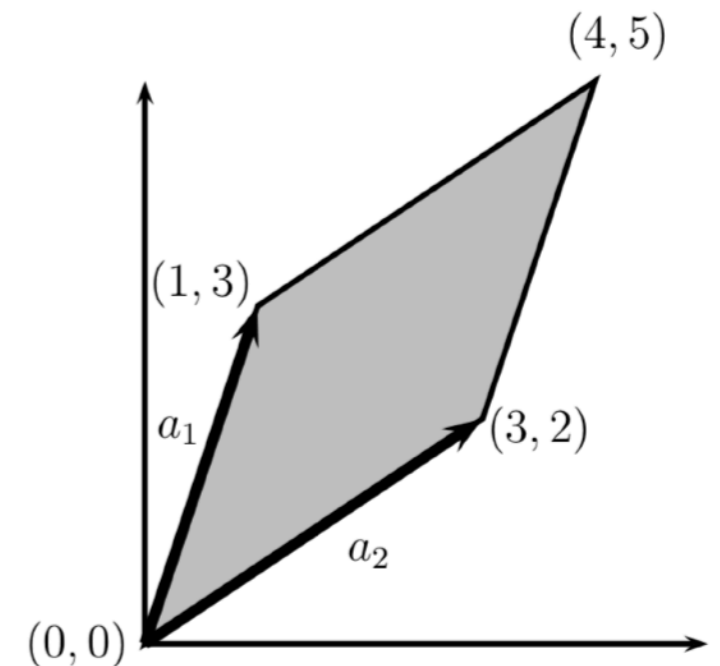
Given a square matrix  $A \in \mathbb{R}^{n \times n}$ , we say that  $\lambda \in \mathbb{C}$  is an *eigenvalue* of  $A$  and  $x \in \mathbb{C}^n$  is the corresponding *eigenvector*<sup>4</sup> if

$$Ax = \lambda x, \quad x \neq 0.$$

- The determinant of  $A$  is equal to the product of its eigenvalues,

$$|A| = \prod_{i=1}^n \lambda_i.$$

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}.$$



## - Diagonalizable Matrices

- Definition: For a matrix  $A \in R^{n \times n}$ , if  $A$  has  $n$  linearly independent eigenvectors, then  $A$  is said to be diagonalizable. In other words,

$$A = UDU^{-1}$$

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From CS357 spring 2019

$$\begin{array}{l}
 \underline{A} \underline{u}_1 = \lambda \underline{u}_1 \\
 \underline{A} \underline{u}_2 = \lambda \underline{u}_2 \\
 \vdots \\
 \underline{A} \underline{u}_n = \lambda \underline{u}_n
 \end{array}
 \Rightarrow
 \begin{array}{c}
 \left[ \begin{array}{c} \underline{A} \\ n \times n \end{array} \right]
 \left[ \begin{array}{c|c|c|c} \underline{u}_1 & \underline{u}_2 & \dots & \underline{u}_n \\ \hline \hline \hline \hline \end{array} \right]
 = \left[ \begin{array}{c|c|c|c} \lambda \underline{u}_1 & \lambda_2 \underline{u}_2 & \dots & \lambda_n \underline{u}_n \\ \hline \hline \hline \hline \end{array} \right] \\
 \\
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 \left[ \begin{array}{c|c|c|c} \lambda_1 & \emptyset & & \\ \hline \emptyset & \lambda_2 & & \\ \hline & & \ddots & \\ \hline & & & \lambda_n \end{array} \right]
 \end{array}$$

$$\underline{A} \underline{U} = \underline{U} \underline{D} \rightarrow \boxed{A = \underline{U} \underline{D} \underline{U}^{-1}}$$

Similarity transformation of  $\underline{A}$

$U^{-1}$  exists  $\Leftarrow$  Note:  $U$  is non-singular since eigenvectors are linearly indep.

## - Quiz

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## - Matrix Singular Value Decomposition

- Definition: factorization a real or complex matrix into three matrices.
- $A = U\Sigma V^T$ , where U and V are orthogonal matrices, and  $\Sigma$  are diagonal matrix.

- How to calculate the decomposed matrices?

$$A = U\Sigma V^T, A^T = V\Sigma U^T$$

$$A^T A = V\Sigma^T U^T U \Sigma V^T$$

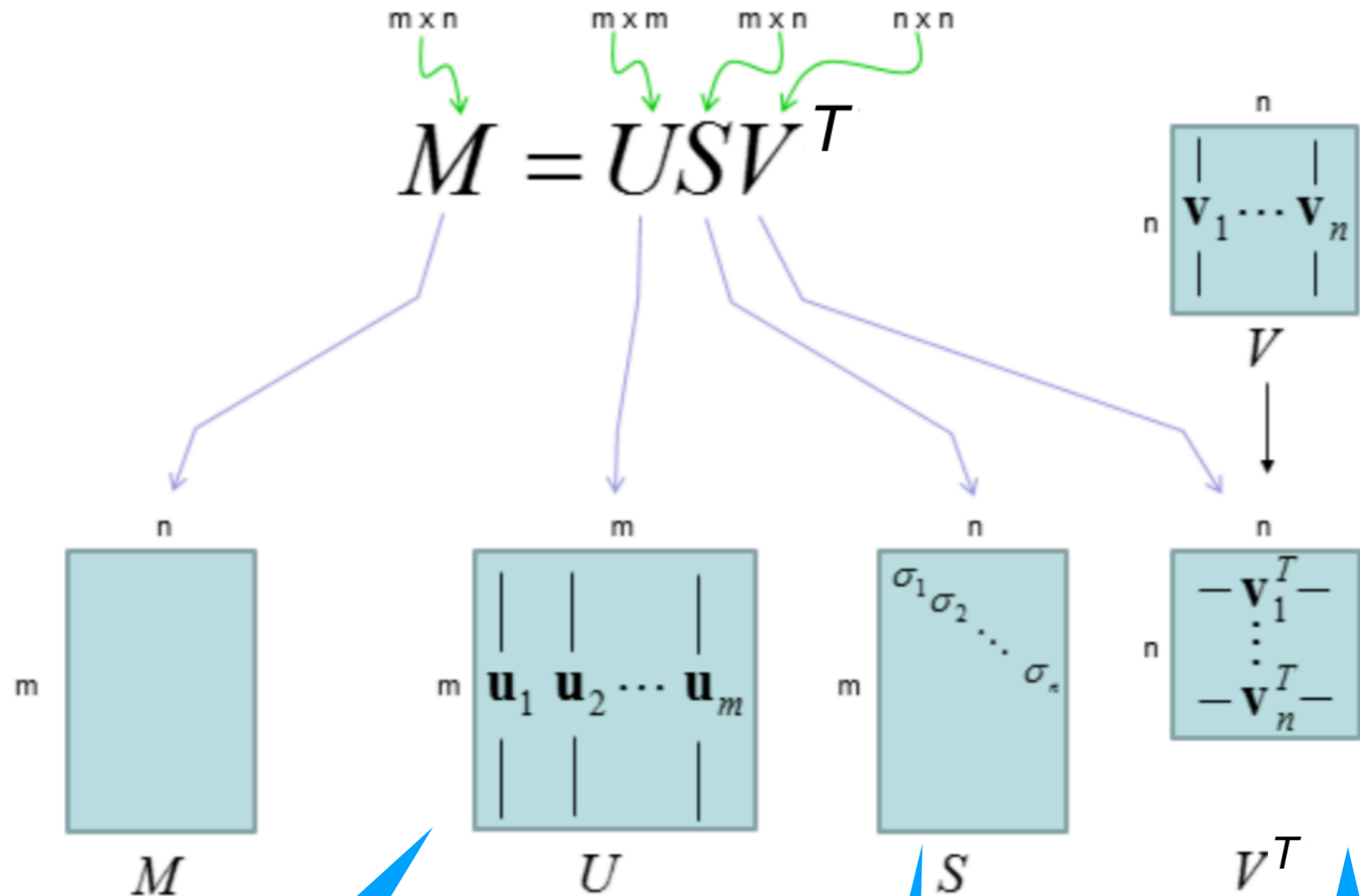
$$A^T A = V\Sigma^T \Sigma V^T$$

$$\text{Let } D = \Sigma^T \Sigma$$

$$A^T A = V D V^T \text{ (diagonalization),}$$

It is in the form of diagonalization! It then indicates that  $V$  is the eigenvectors of  $A^T A$ , and the square root of  $D$  gives us the value of  $\Sigma$

- Matrix Singular Value Decomposition (not reduced)



The columns of  $U$  are the eigenvectors of  $MM^T$

The diagonals of  $S$  are the square root of eigenvalues of  $M^T M$

The columns of  $V$  are the eigenvectors of  $M^T M$