Announcements

• Midterm Exam
• 6:30 – 7:45 PM Wed. March 1

• A-M If the first letter of your NetID is A-M
  take the exam here 1320DCL
• N-Z If the first letter of your NetId is N-Z
  take the exam Loomis 151 (Physics building)

• We only have just enough seats so you MUST take it in the assigned room
• We will not grade exams that are handed in from the wrong room
Weak & Strong LLN

Let $\overline{c_n}$ be the average of $n$ samples from “reasonable” random variable $C$ (well defined mean, bounded variance)

$$\lim_{n \to \infty} Pr(|\overline{c_n} - E[C]| > \varepsilon) = 0$$

$$Pr(\lim_{n \to \infty} \overline{c_n} = E[C]) = 1$$
Some Distributions

- Normal (aka Gaussian)
- Binomial
- Beta
- Multinomial / Dirichlet
- Moments (for well-behaved distributions)
  - first moment = mean $\mu$
  - second central moment = variance $\sigma^2$ ("dispersion about the mean")
  - sometimes more convenient: standard deviation $\sigma$ (units)
  - skew, kurtosis
- Discrete probability mass function
- Continuous probability density function
  (integrate over a region to get a probability)
- Support (domain where nonzero)
Normal Distribution (aka Gaussian)

\[ X \sim N(\mu, \sigma^2) \]

Continuous
Location \( \mu \)
Shape \( \sigma^2 \)
Support \((-\infty, +\infty)\)

\[
Pr(x; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

When estimated from sampled data, many distributions from the exponential family approach Normal as \(|\text{samples}|\) increases.
Binomial Distribution

\[ X \sim \text{Bin}(p, n) \]

\[ Pr(x; p, n) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n \]

Recall
\[ \binom{n}{x} = \frac{n!}{x!(n-x)!} \]

**Discrete**

**Mean** \( np \)

**Variance** \( np(1-p) \)

**Support** \([0, n]\)

Discrete & Finite support but... looks quite Normal w/ 100 samples
Beta Distribution

• N flips of a coin with weight p
• Of the N, h are heads and t are tails
• So N = h + t
• Given p and N what are the distributions of h and t?
• Then h ~ Binomial(p, N)
  t ~ Binomial(1-p, N)
• h, t, and p are related
• Given h and t, (recall N = h + t)
• Then p ~ Beta(h+1, t+1)
• p ~ Beta(a, b) (where a=h+1 and b=t+1)
• a, b > 0
• The standard Beta has support [0,1]
Beta Distribution

- Beta(a, b) \(a, b > 0\)
- Think about a & b as abstract shape parameters (not integer numbers of head & tails off by 1)
- Mean: \(E(Beta(a, b)) = \frac{a}{a + b}\)
- Variance: \(Beta(a, b) = \frac{ab}{(a+b)(a+b)(a+b+1)}\)
- Consider Beta for fixed \(a / (a + b)\)
- Expected coin weighting is the mean: \(a / (a + b)\)
- What is the probability that the true mean is \(a / (a + b)\)?
- Confidence that the true mean lies within \(\pm \varepsilon\) of the expected value is the area under Beta pdf integrating \(\pm \varepsilon\) about \(a / (a + b)\)
- \textit{Conjugate prior} to Binomial
  - Use Beta as a prior for Binomial’s p
  - See some data
  - Posterior on p is also Beta
Beta Distribution

As $a+b$ increases, the distribution becomes more peaked at its mean and once again approaches Normal.

The shape becomes close to Normal. The support is $[0,1]$ but the low and high tails of the Normal represent only a tiny fraction of the probability mass.
Any Interpretation for Beta(0.1, 0.1)?

Strong prior belief that the coin came from a magic store, not the mint.

But we do not whether it is weighted for heads or weighted for tails
Multinomial / Dirichlet

• Multivariate analogs of Binomial / Beta
• Multinomial
  – instead of a coin
  – say a six-sided die
  – $N$ and $(p_1, p_2, p_3, p_4, p_5, p_6)$ where $\sum = 1$
• Dirichelet
  – shape parameters $(a_1, a_2, a_3, a_4, a_5, a_6)$ each $> 0$
  – $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 1$ Uniform
  – with $0 < a_i < 1$ bias toward sparsity
Central Limit Theorem

• Compute the average of $n$ samples from a random variable
• This is an estimate of the mean of the R.V.
• Repeat this many times, each time with $n$ new independent samples
• The estimated mean is also a random variable
• Its distribution approaches Normal as $n$ increases
• (Provided the original R.V. has a reasonable distribution)
• We can even mix different R.V.s following a fixed random scheme
How Do We Use These Distributions?

Recall our Fun / Weather Joint

<table>
<thead>
<tr>
<th>Weather</th>
<th>Sunny</th>
<th>Cloudy</th>
<th>Rainy</th>
<th>Snowy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Have Fun? Yes</td>
<td>0.25</td>
<td>0.15</td>
<td>0.05</td>
<td>0.13</td>
</tr>
<tr>
<td>No</td>
<td>0.05</td>
<td>0.1</td>
<td>0.25</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Could we use a Gaussian instead?

Probably not, our joint is more appropriate
Gaussian: 2DOF; Table: 7DOF
Recall our Q Tables

<table>
<thead>
<tr>
<th>Actions</th>
<th>States</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s_1$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>32.7</td>
</tr>
<tr>
<td>$a_2$</td>
<td>...</td>
</tr>
<tr>
<td>$a_3$</td>
<td>...</td>
</tr>
<tr>
<td>$a_4$</td>
<td>16.9</td>
</tr>
<tr>
<td>$a_5$</td>
<td>12.2</td>
</tr>
</tbody>
</table>

With optimal policy $\pi(s) = \arg \max_a Q(a, s)$
Recall our Q Tables

<table>
<thead>
<tr>
<th>Actions</th>
<th>States</th>
<th>s₁</th>
<th>s₂</th>
<th>s₃</th>
<th>s₄</th>
<th>s₅</th>
<th>s₆</th>
<th>s₇</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁</td>
<td></td>
<td>32.7</td>
<td>13.6</td>
<td>-5.8</td>
<td>10.3</td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₂</td>
<td></td>
<td></td>
<td>2.5</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₃</td>
<td></td>
<td></td>
<td>-9.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

CLT justifies modeling Q entries as estimated means of Normals

Implications for exploration: a₅ vs a₁ and a₅ vs a₃
End of Midterm Material
Bayesian Networks (belief networks,...)

- Represents general probability distributions
- Graphical model
  - Probability + Graph theory
- Nodes – discrete random variables
- Directed Arcs – direct influence
  - Usually “causality” (class, text,...)
- Configuration of parents establishes contexts for a node
  - Different distribution for each context
  - Conditional Probability Tables (CPTs)
- No directed cycles (DAG)
- General graphical models
  - Markov Networks
  - Conditional random fields
  - Dynamic Bayesian nets
  - Plate models...