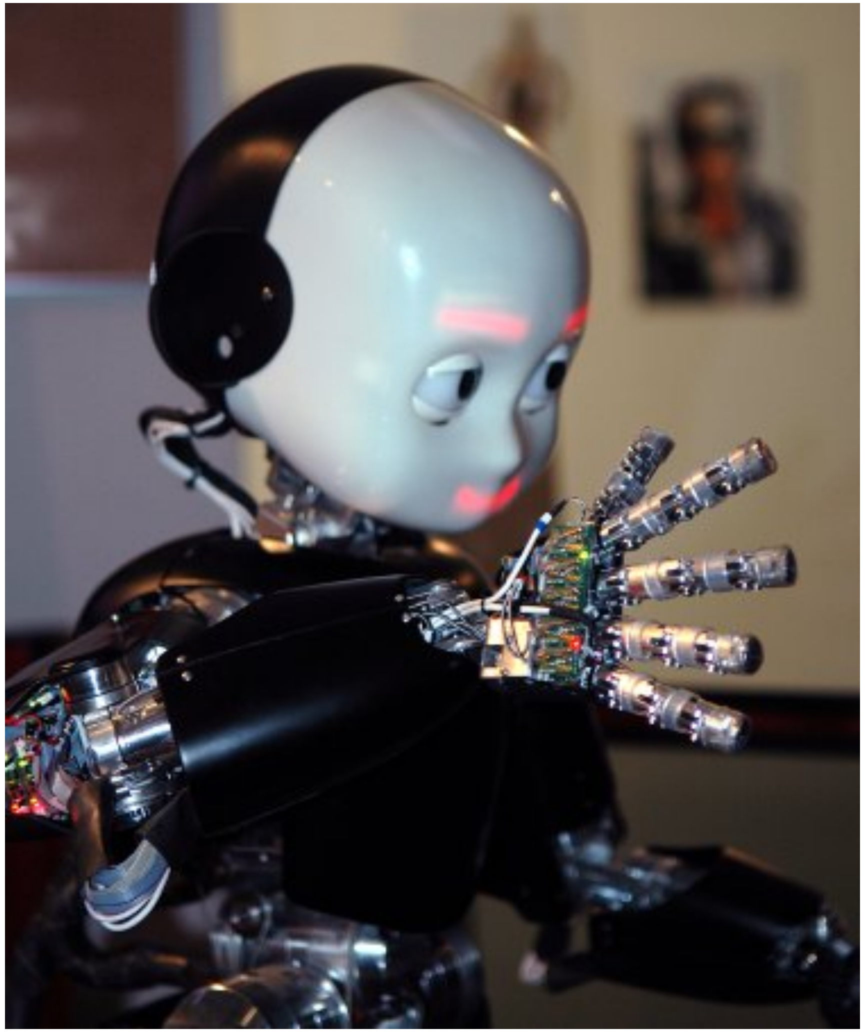
# CS440/ECE448 Lecture 21: Markov Decision Processes

Slides by Svetlana Lazebnik, 11/2016

Modified by Mark Hasegawa-Johnson, 3/2019



# Markov Decision Processes

- In HMMs, we see a **sequence of observations** and try to reason about the **underlying state sequence**
  - There are no actions involved
- But what if we have to take an **action** at each step that, in turn, will **affect the state of the world**?

# Markov Decision Processes (MDPs)

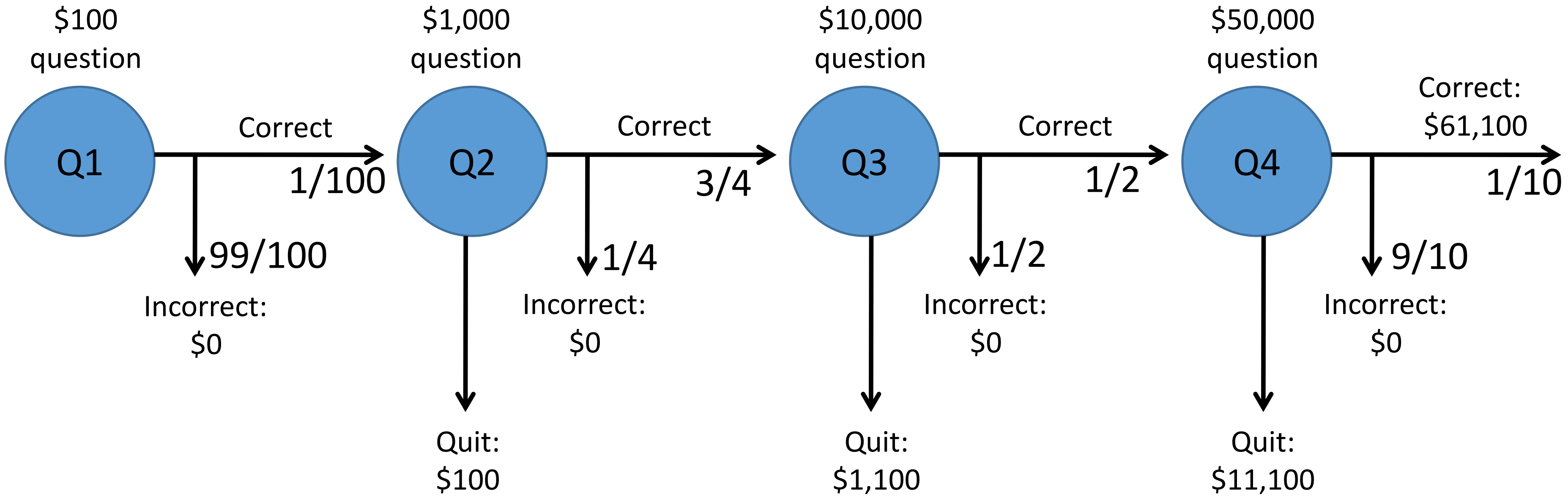
- Components that define the MDP. Depending on the problem statement, you either know these, or you learn them from data:
  - **States** $s$, beginning with initial state $s_0$
  - **Actions** $a$
    - Each state $s$ has actions $A(s)$ available from it
  - **Transition model** $P(s' \mid s, a)$
    - *Markov assumption*: the probability of going to $s'$ from $s$ depends only on $s$ and $a$ and not on any other past actions or states
  - **Reward function** $R(s)$
- **Policy – the "solution" to the MDP:**
  - $\pi(s) \in A(s)$: the action that an agent takes in any given state

# Overview

- First, we will look at how to "solve" MDPs, or **find the optimal policy** when the transition model and the reward function are known
- Second, we will consider **reinforcement learning**, where we don't know the rules of the environment or the consequences of our actions

# Game show

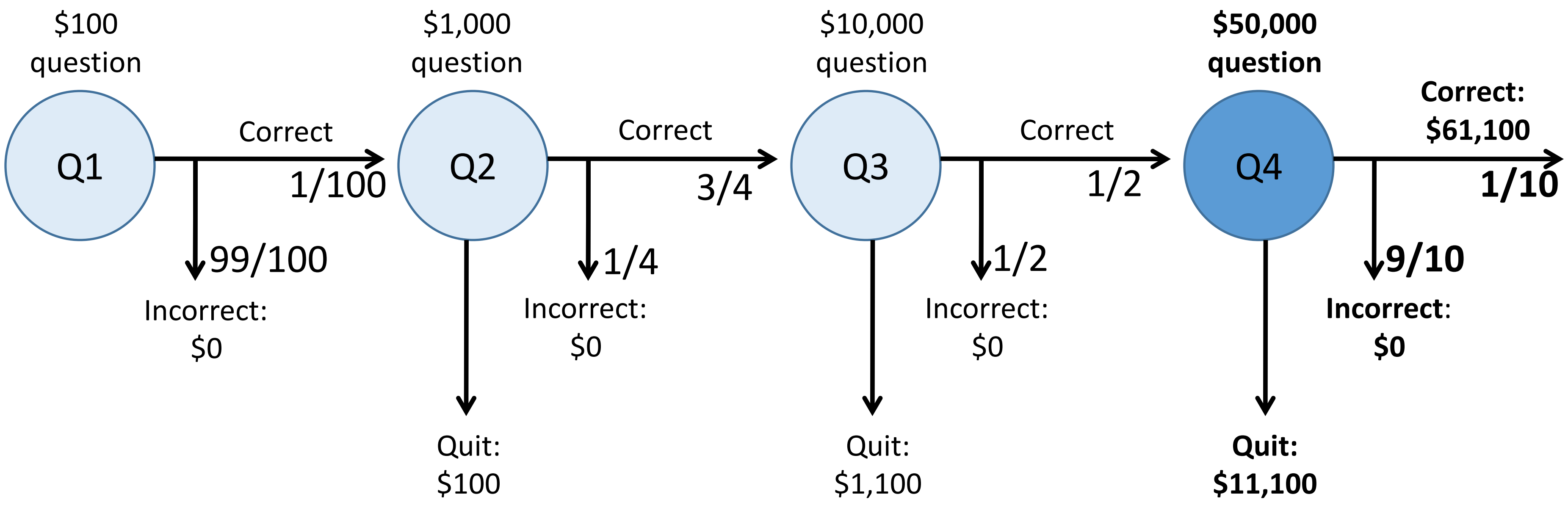
- A series of questions with increasing level of difficulty and increasing payoff
- Decision: at each step, take your earnings and quit, or go for the next question
  - If you answer wrong, you lose everything



# Game show

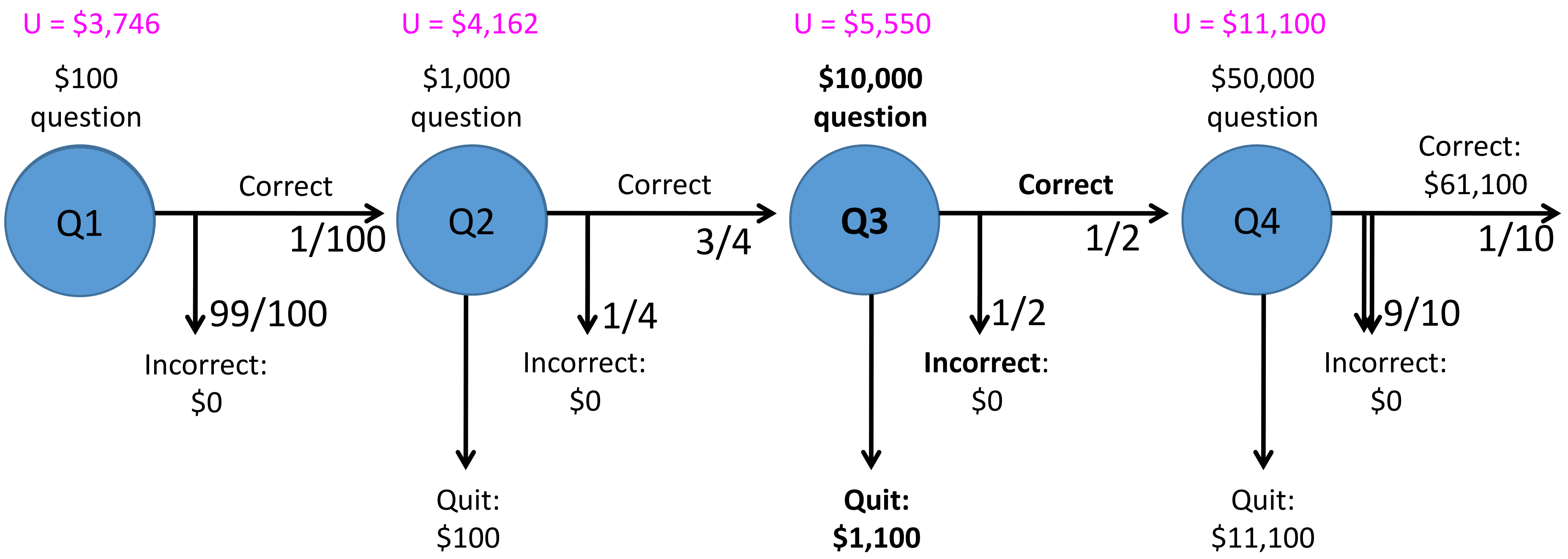
- Consider the $50,000 question
  - Probability of guessing correctly: 1/10
  - Quit or go for the question?
- What is the expected payoff for continuing?

  0.1 * 61,100 + 0.9 * 0 = 6,110
- What is the optimal decision?

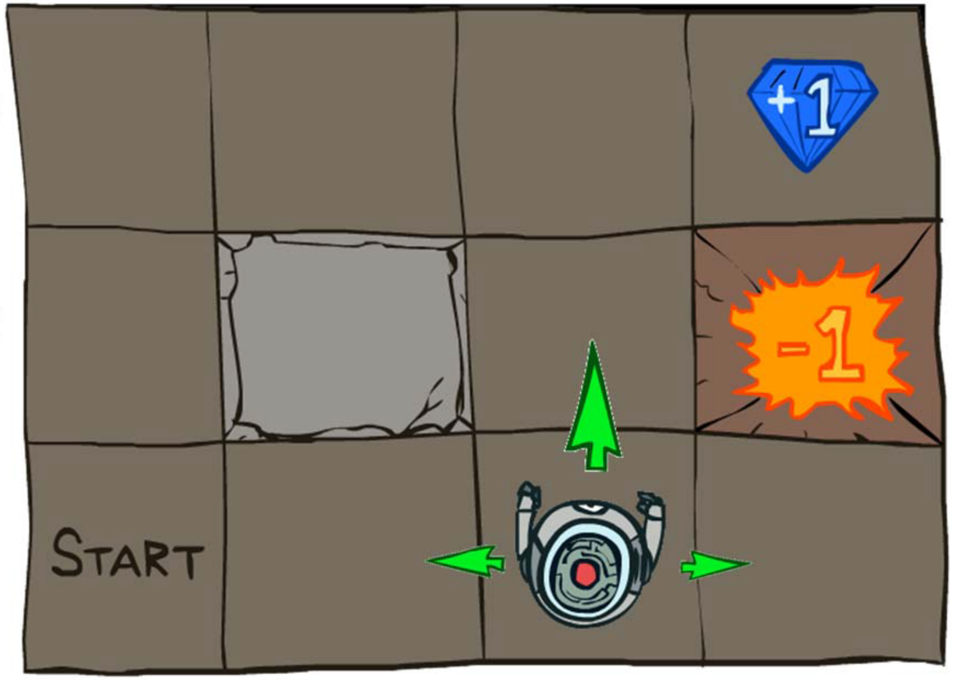


# Game show

- What should we do in Q3?
  - Payoff for quitting: $1,100
  - Payoff for continuing: 0.5 * $11,100 = $5,550
- What about Q2?
  - $100 for quitting vs. $4,162 for continuing
- What about Q1?

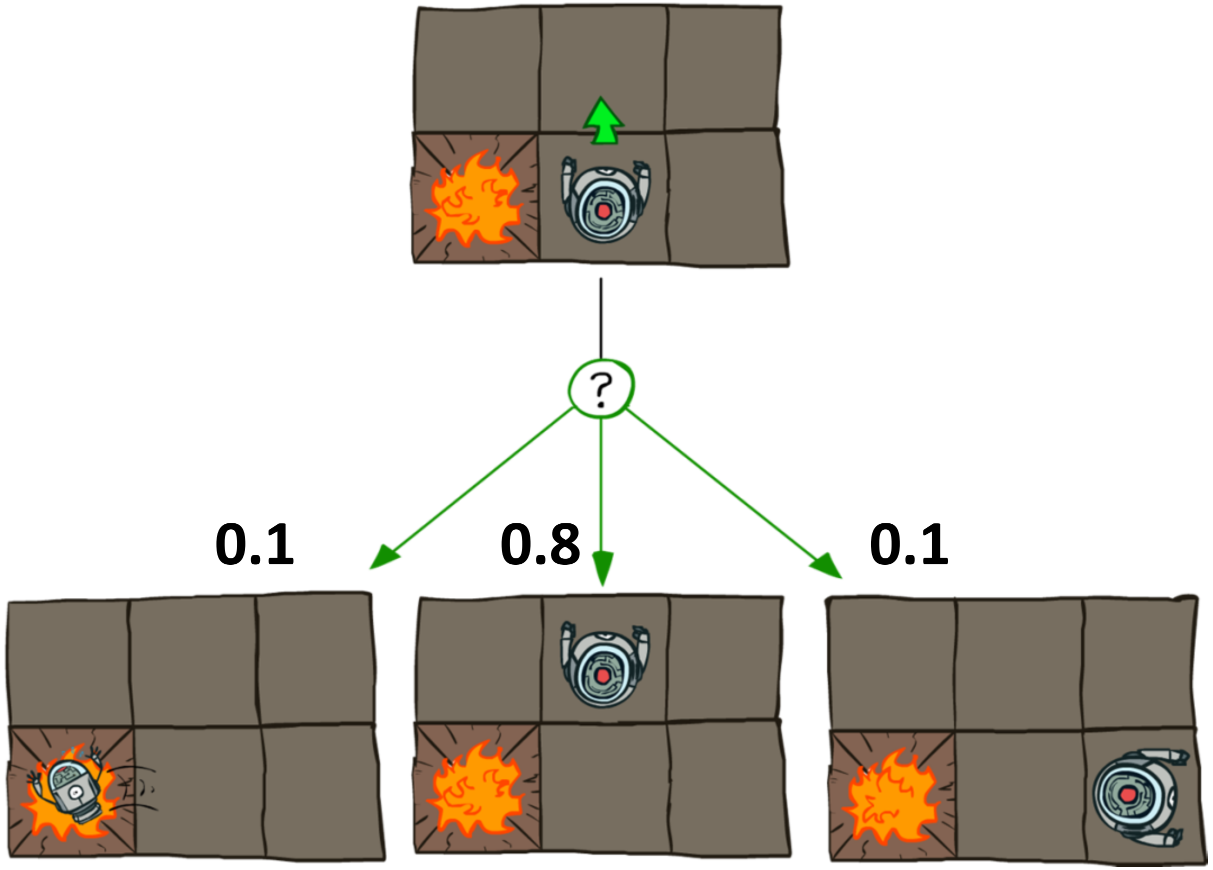


# Grid world



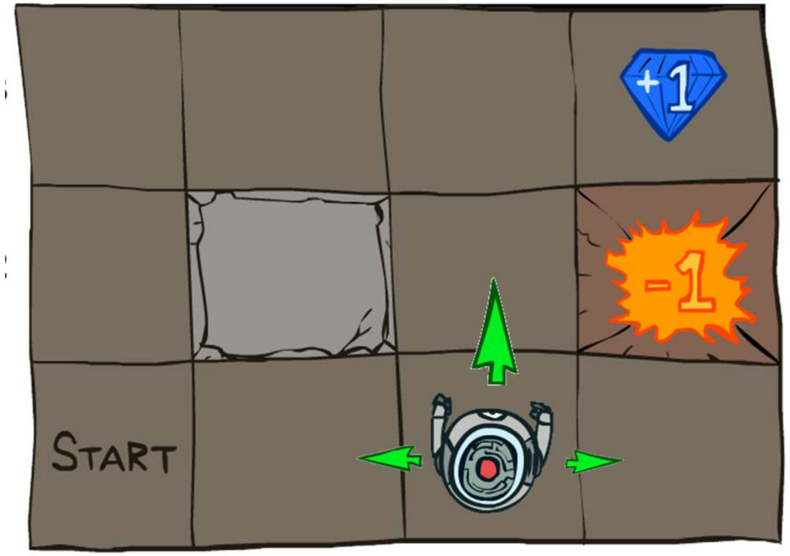

R(s) = -0.04 for every non-terminal state

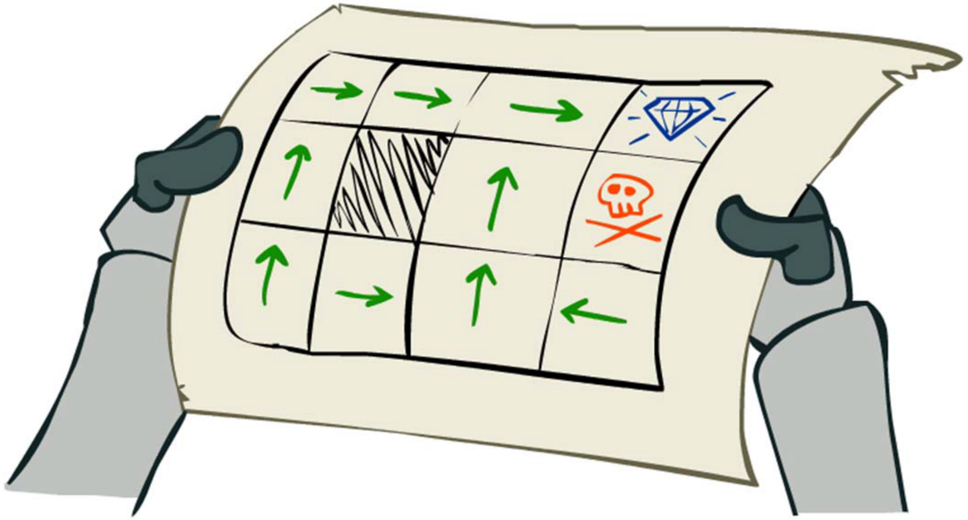
Transition model:




Source: P. Abbeel and D. Klein

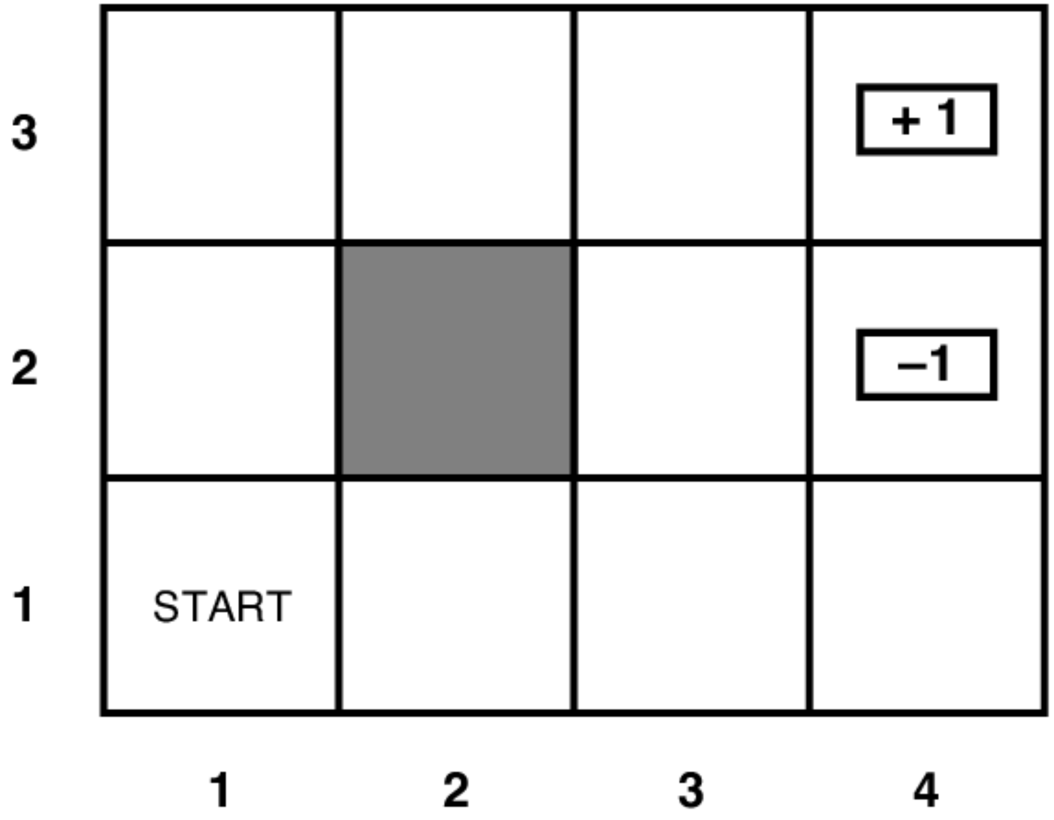
# Goal: Policy



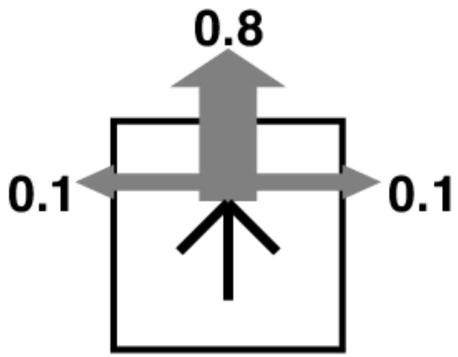



Source: P. Abbeel and D. Klein

# Grid world

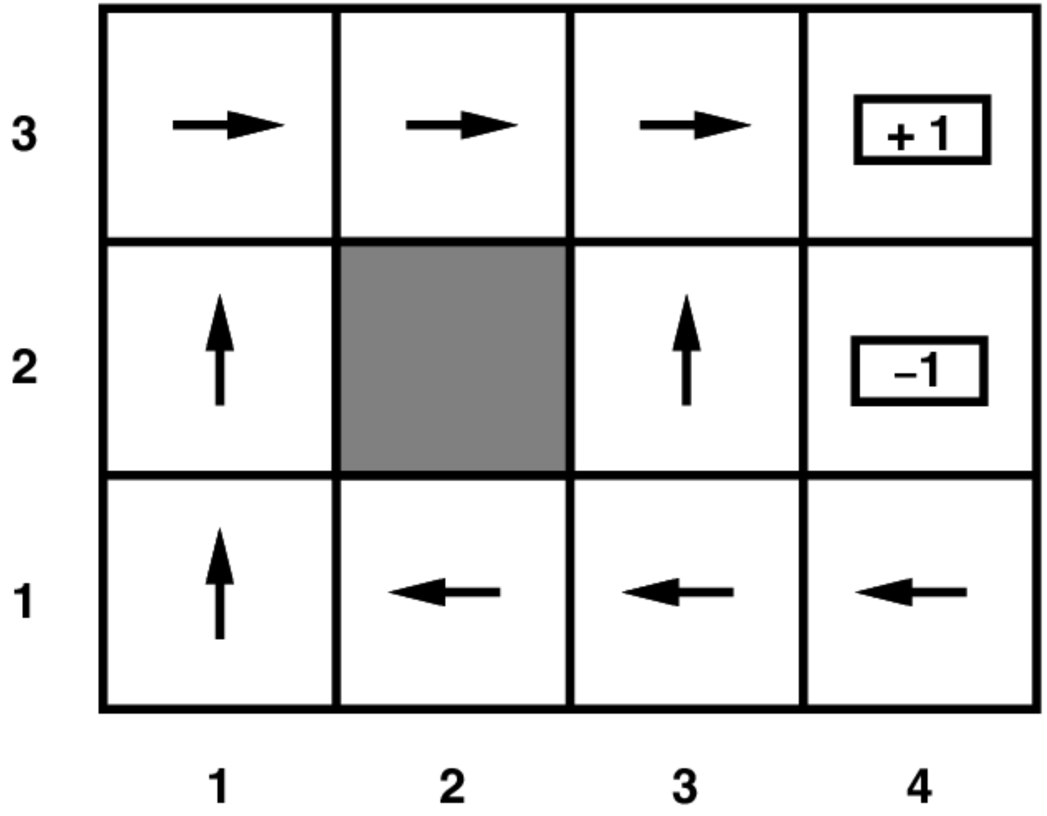



Transition model:




R(s) = -0.04 for every non-terminal state

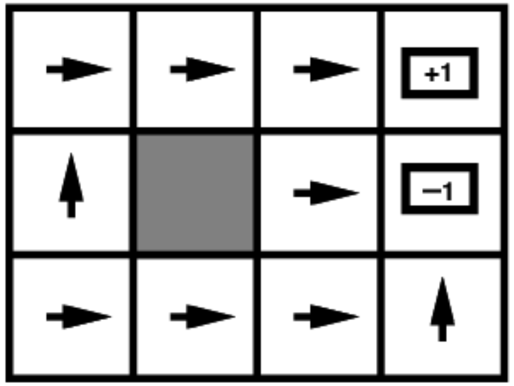
# Grid world



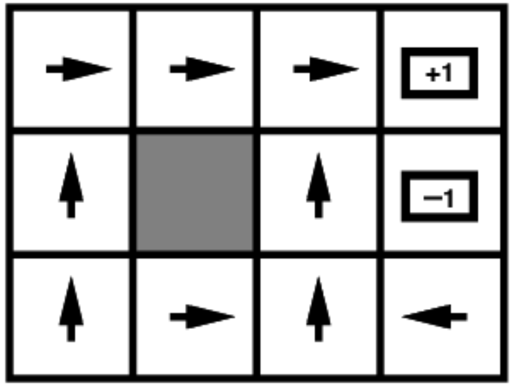

Optimal policy when R(s) = -0.04 for every non-terminal state

# Grid world

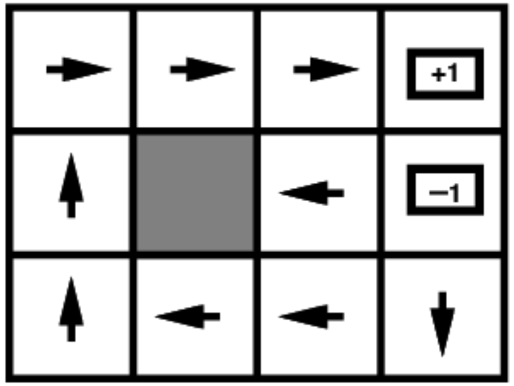
- Optimal policies for other values of R(s):



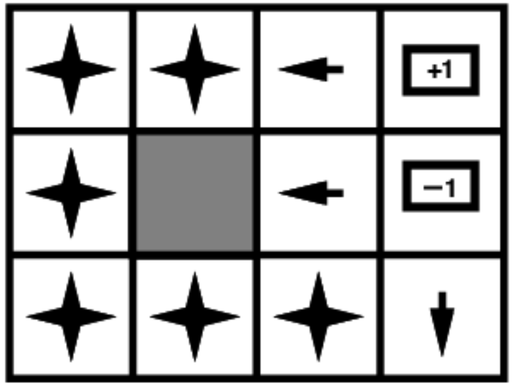

$R(s) < -1.6284$




$-0.4278 < R(s) < -0.0850$




$-0.0221 < R(s) < 0$




$R(s) > 0$

# Solving MDPs

- MDP components:
  - **States** $s$
  - **Actions** $a$
  - **Transition model** $P(s' \mid s, a)$
  - **Reward function** $R(s)$
- The solution:
  - **Policy** $\pi(s)$: mapping from states to actions
  - How to find the optimal policy?

# Maximizing expected utility

- The **optimal policy** π(s) should **maximize the *expected utility* over all possible state sequences** produced by following that policy:

$$\sum_{\substack{state\ sequences \\ starting\ from\ s_0}} P\big(sequence \mid s_0, a = \pi(s_0)\big)\, U(sequence)$$

- How to define the **utility of a state sequence**?
  - **Sum of rewards** of individual states
  - Problem: **infinite state sequences**

# Utilities of **state sequences**

- Normally, we would define the utility of a state sequence as the sum of the rewards of the individual states
- **Problem:** infinite state sequences
- **Solution:** *discount* the individual state rewards by a factor $\gamma$ between 0 and 1:

$$U([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots$$

$$= \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \frac{R_{\max}}{1-\gamma} \qquad (0 < \gamma < 1)$$

  - Earlier rewards count more than later rewards
  - Makes sure the total utility stays bounded
  - Helps algorithms converge

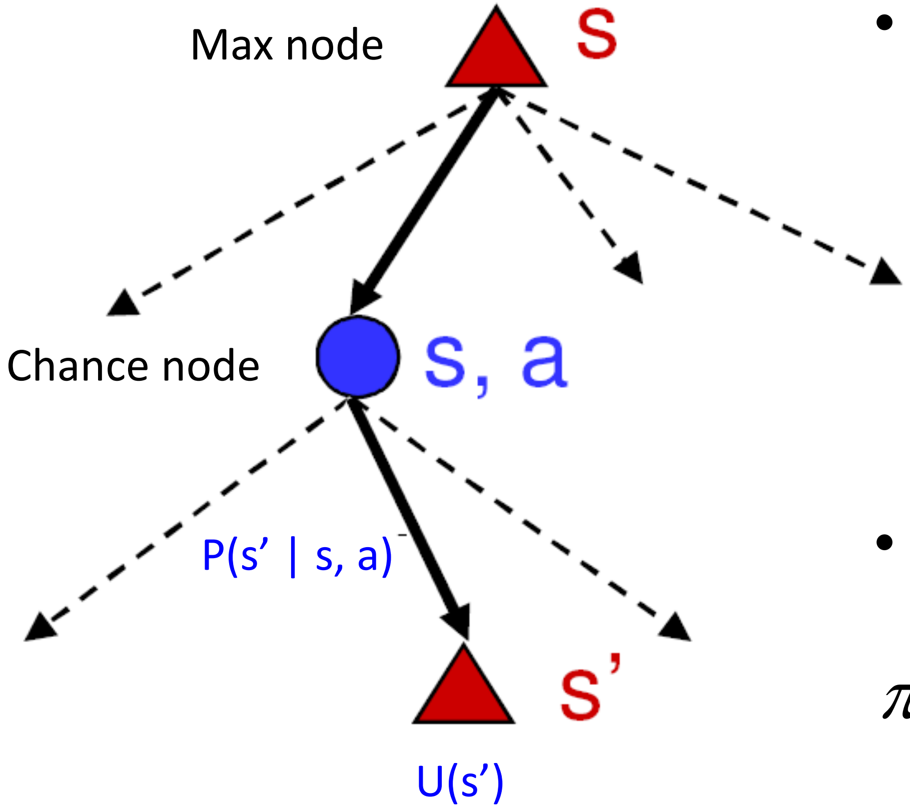
# Utilities of **states**

- **Expected utility obtained by policy π starting in state s**:

$$U^{\pi}(s) = \sum_{\substack{state\ sequences \\ starting\ from\ s}} P\big(sequence|s, a = \pi(s)\big)U(sequence)$$

- The **"true" utility of a state**, denoted U(s), is the *best possible* expected sum of discounted rewards
    - if the agent executes the *best possible* policy starting in state s
- Reminiscent of minimax values of states…

# Finding the utilities of **states**




- If state s' has utility U(s'), then what is the expected utility of taking action **a** in state **s**?

$$\sum_{s'} P(s' \mid s, a) U(s')$$

- How do we choose the optimal action?

$$\pi^*(s) = \arg\max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')$$

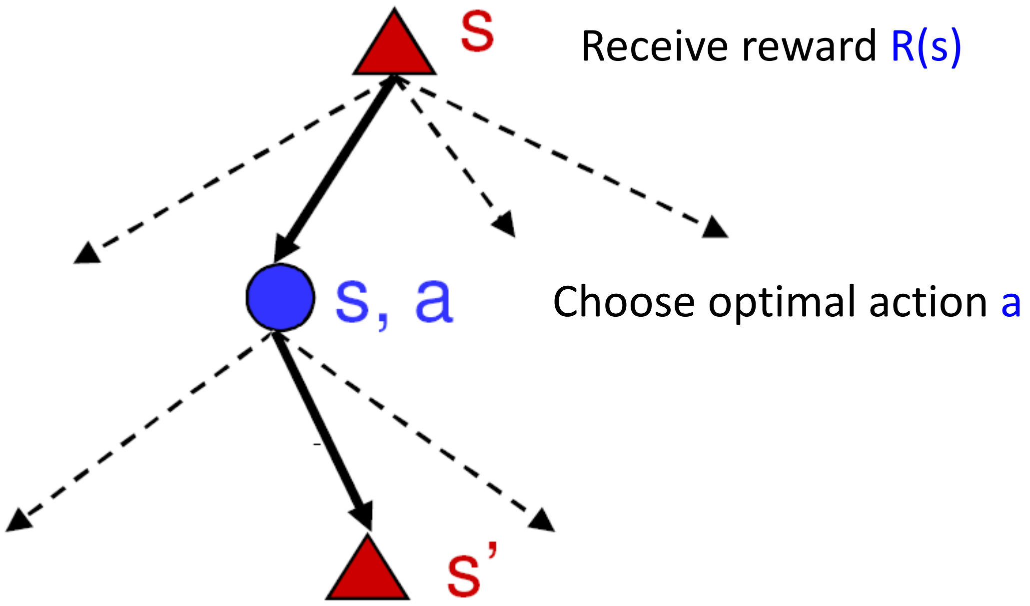
- What is the recursive expression for **U(s)** in terms of the utilities of its successor states?

$$U(s) = R(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) U(s')$$

# The Bellman equation

- Recursive relationship between the utilities of successive states:

$$\boxed{U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')}$$




End up here with $P(s' \mid s, a)$
Get utility $U(s')$
(discounted by $\gamma$)

# The Bellman equation

- Recursive relationship between the utilities of successive states:

$$\boxed{U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')}$$

- For *N* states, we get *N* equations in *N* unknowns
  - Solving them solves the MDP
  - Nonlinear equations -> no closed-form solution, need to use an iterative solution method (is there a globally optimum solution?)
  - We could try to solve them through expectiminimax search, but that would run into trouble with infinite sequences
  - Instead, we solve them algebraically
  - Two methods: **value iteration** and **policy iteration**

# Method 1: Value iteration

- Start out with every $U(s) = 0$
- Iterate until convergence
  - During the $i$th iteration, update the utility of each state according to this rule:

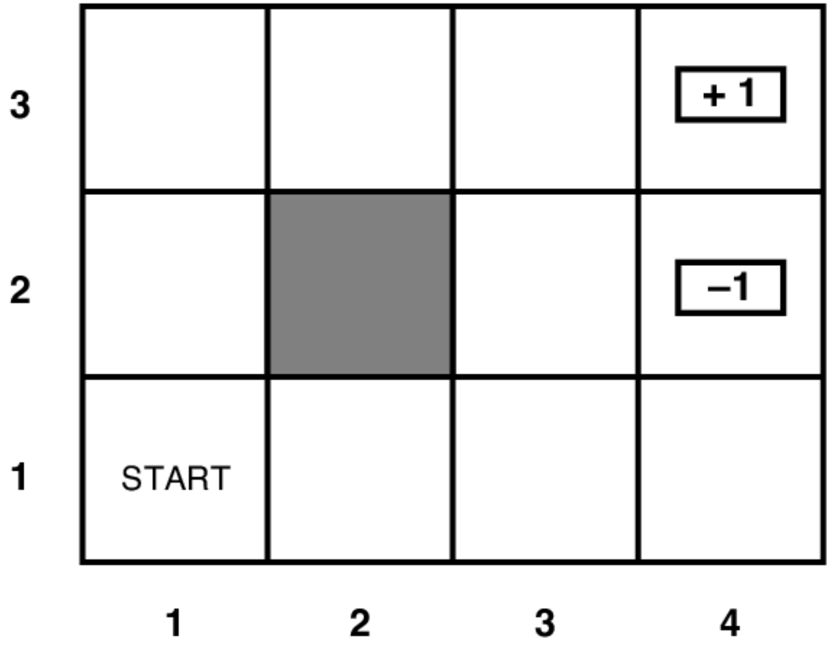
$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' | s, a) U_i(s')$$

- In the limit of infinitely many iterations, this is guaranteed to find the correct utility values
  - Error decreases exponentially, so in practice, don't need an infinite number of iterations…

# Value iteration

- What effect does the update have?

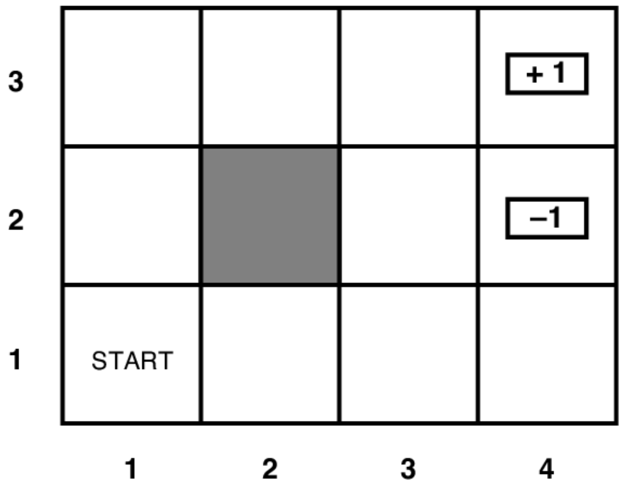
$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' | s, a) U_i(s')$$

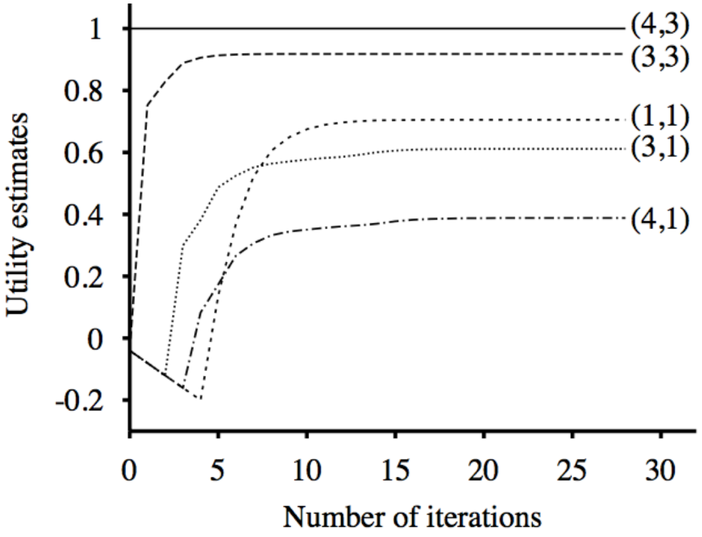



[Value iteration demo]

# Value iteration

Input (non-terminal R=-0.04)

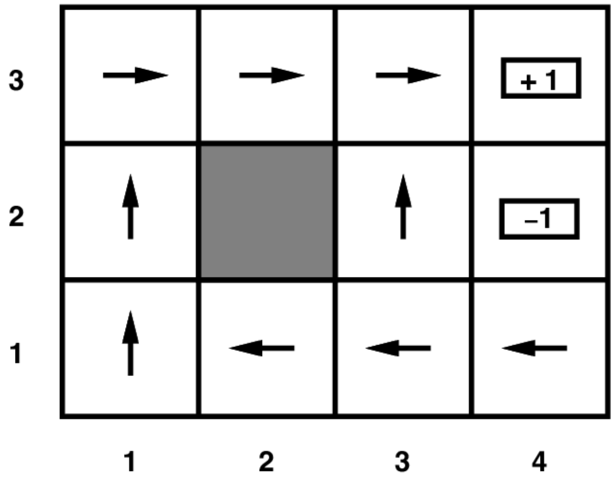



Utilities with discount factor 1

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 3 | 0.812 | 0.868 | 0.918 | +1 |
| 2 | 0.762 | | 0.660 | –1 |
| 1 | 0.705 | 0.655 | 0.611 | 0.388 |

Final policy



# Method 2: Policy iteration

- Start with some **initial policy** $\pi_0$ and alternate between the following steps:
  - **Policy evaluation:** calculate $U^{\pi_i}(s)$ for every state $s$
  - **Policy improvement:** calculate a new policy $\pi_{i+1}$ based on the updated utilities

- Notice it's kind of like **hill-climbing** in the N-queens problem.
  - **Policy evaluation**: Find ways in which the current policy is suboptimal
  - **Policy improvement**: Fix those problems

- Unlike Value Iteration, this is guaranteed to **converge in a *finite* number of steps**, as long as the **state space and action set are both finite**.

# Method 2, Step 1: **Policy evaluation**

- Given a **fixed policy** $\pi$, calculate $U^\pi(s)$ for every state $s$

$$\boxed{U^\pi(s) = R(s) + \gamma \sum_{s'} P(s' | s, \pi(s)) U^\pi(s')}$$

- $\pi$(s) is fixed, therefore $P(s'|s,\pi(s))$ is an $s' \times s$ matrix, therefore we can solve a linear equation to get $U^\pi(s)$!
- Why is this "Policy Evaluation" formula so much easier to solve than the original Bellman equation?

$$\boxed{U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' | s, a) U(s')}$$

# Method 2, Step 2: Policy improvement

- Given $U^{\pi}(s)$ for every state $s$, find an **improved** $\pi(s)$

$$\pi^{i+1}(s) = \arg\max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U^{\pi_i}(s')$$

# Summary

- MDP defined by **states, actions, transition model, reward function**
- **The "solution" to an MDP is the policy**:
  what do you do when you're in any given state
- The **Bellman equation** tells the **utility of any given state**, and incidentally, also tells you the **optimum policy**.
  The Bellman equation is **N nonlinear equations in N unknowns** (the policy), therefore it can't be solved in closed form.
- **Value iteration:**
  - At the beginning of the (i+1)'st iteration, each state's value is based on looking ahead i steps in time
  - … so finding the best action = optimize based on (i+1)-step lookahead
- **Policy iteration:**
  - Find the utilities that result from the current policy,
  - Improve the current policy