CS 440/ECE448 Lecture 19: Bayes Net Inference

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Including slides by Svetlana Lazebnik, 11/2016
Bayes Network Inference & Learning

Bayes net is a **memory-efficient model** of dependencies among:
- Query variables: \( X \)
- *Evidence (observed)* variables and their values: \( E = e \)
- *Unobserved* variables: \( Y \)

**Inference problem**: answer questions about the query variables given the evidence variables
- This can be done using the posterior distribution \( P(X \mid E = e) \)
- The posterior can be derived from the full joint \( P(X, E, Y) \)
- How do we make this **computationally efficient**?

**Learning problem**: given some training examples, how do we learn the parameters of the model?
- Parameters = \( p(\text{variable} \mid \text{parents}) \), for each variable in the net
Outline

• Inference Examples
• Inference Algorithms
  • Trees: Sum-product algorithm
  • Poly-trees: Junction tree algorithm
  • Graphs: No polynomial-time algorithm
• Parameter Learning
Practice example 1

- Variables: *Cloudy, Sprinkler, Rain, Wet Grass*
Practice example 1

• Given that the grass is wet, what is the probability that it has rained?

\[
P(r \mid w) = \frac{P(r, w)}{P(w)} = \frac{\sum_{C=c, S=s} P(c, s, r, w)}{\sum_{C=c, S=s, R=r} P(c, s, r, w)}
\]

\[
= \sum_{C=c, S=s} P(c)P(s \mid c)P(r \mid c)P(w \mid r, s)
\]

\[
= \frac{\sum_{C=c, S=s} P(c)P(s \mid c)P(r \mid c)P(w \mid r, s)}{\sum_{C=c, S=s, R=r} P(c)P(s \mid c)P(r \mid c)P(w \mid r, s)}
\]
Practice Example #2

• Suppose you have an observation, for example, “Jack called” (J=1)
• You want to know: was there a burglary?
• You need

\[
P(B | J = 1) = \frac{P(B, J = 1)}{\sum_j P(B, J = j)}
\]

• So you need to compute the table \(P(B,J)\) for all possible settings of \((B,J)\)
Bayes Net Inference: The Hard Way

1. \( P(B, E, A, J, M) = P(B)P(E)P(A|B,E)P(J|A)P(M|A) \)

2. \( P(B, J) = \sum_E \sum_A \sum_M P(B, E, A, J, M) \)

Exponential complexity (\#P-hard, actually): \( N \) variables, each of which has \( K \) possible values \( \Rightarrow O\{K^N\} \) time complexity
Is there an easier way?

• Tree-structured Bayes nets: the sum-product algorithm
  • Quadratic complexity, $O\{NK^3\}$
• Polytrees: the junction tree algorithm
  • Pseudo-polynomial complexity, $O\{NK^M\}$, for $M<N$
• Arbitrary Bayes nets: #P complete, $O\{K^N\}$
  • The SAT problem is a Bayes net!
• Parameter Learning
1. Tree-Structured Bayes Nets

- Suppose these are all binary variables.
- We observe E=1
- We want to find P(H=1|E=1)
- Means that we need to find both P(H=0,E=1) and P(H=1,E=1) because

\[
P(H = 1|E = 1) = \frac{P(H = 1, E = 1)}{\sum_h P(H = h, E = 1)}
\]
The Sum-Product Algorithm (Belief Propagation)

- Find the only undirected path from the evidence variable to the query variable (EDBFGIH)
- Find the directed root of this path $P(F)$
- Find the joint probabilities of root and evidence: $P(F=0,E=1)$ and $P(F=1,E=1)$
- Find the joint probabilities of query and evidence: $P(H=0,E=1)$ and $P(H=1,E=1)$
- Find the conditional probability $P(H=1|E=1)$
The Sum-Product Algorithm (Belief Propagation)

Starting with the root $P(F)$, we find $P(F,E)$ by alternating product steps and sum steps:

1. Product: $P(B,D,F) = P(F)P(B|F)P(D|B)$
2. Sum: $P(D,F) = \sum_{B=0}^{1} P(B,D,F)$
3. Product: $P(D,E,F) = P(D,F)P(E|D)$
4. Sum: $P(E,F) = \sum_{D=0}^{1} P(D,E,F)$
The Sum-Product Algorithm (Belief Propagation)

Starting with the root $P(E,F)$, we find $P(E,H)$ by alternating product steps and sum steps:

1. Product: $P(E,F,G)=P(E,F)P(G|F)$
2. Sum: $P(E,G) = \sum_{F=0}^{1} P(E,F,G)$
4. Sum: $P(E,I) = \sum_{G=0}^{1} P(E,G,I)$
6. Sum: $P(E,H) = \sum_{I=0}^{1} P(E,H,I)$
Time Complexity of Belief Propagation

- Each product step generates a table with 3 variables
- Each sum step reduces that to a table with 2 variables
- If each variable has K values, and if there are $O(N)$ variables on the path from evidence to query, then time complexity is $O(NK^3)$
Time Complexity of Bayes Net Inference

- Tree-structured Bayes nets: the sum-product algorithm
  - Quadratic complexity, $O\{NK^3\}$
- Polytrees: the junction tree algorithm
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  - The SAT problem is a Bayes net!
- Parameter Learning
2. The Junction Tree Algorithm

a. Moralize the graph (identify each variable’s Markov blanket)
b. Triangulate the graph (eliminate undirected cycles)
c. Create the junction tree (form cliques)
d. Run the sum-product algorithm on the junction tree
2.a. Markov Blanket

• Suppose there is a Bayes net with variables A,B,C,D,E,F,G,H
• The “Markov blanket” of variable F is D,E,G if
  \[
P(F|A,B,C,D,E,G,H) = P(F|D,E,G)
  \]
2.a. Markov Blanket

- Suppose there is a Bayes net with variables A, B, C, D, E, F, G, H
- The “Markov blanket” of variable F is D, E, G if
  \[ P(F|A, B, C, D, E, G, H) = P(F|D, E, G) \]
2.a. Markov Blanket

• The “Markov blanket” of variable F is D,E,G if
  \[ P(F|A,B,C,D,E,G,H) = P(F|D,E,G) \]
  • How can we prove that?
  • \( P(A,...,H) = P(A)P(B|A) \) ...
  • Which of those terms include F?
2.a. Markov Blanket

• Which of those terms include F?
• Only these two:
  \[ P(F|D) \]
  and
  \[ P(G|E,F) \]
2.a. Markov Blanket

The Markov Blanket of variable F includes only its immediate family members:

• Its parent, D
• Its child, G
• The other parent of its child, E

Because \( P(F|A,B,C,D,E,G,H) = P(F|D,E,G) \)
2.a. Moralization

“Moralization” =

1. If two variables have a child together, force them to get married.
2. Get rid of the arrows (not necessary any more).

Result: Markov blanket = the set of variables to which a variable is connected.
2.b. Triangulation

Triangulation = draw edges so that there is no unbroken cycle of length > 3.

There are usually many different ways to do this. For example, here’s one:
2.c. Form Cliques

Clique = a group of variables, all of whom are members of each other’s immediate family.

Junction Tree = a tree in which
• Each node is a clique from the original graph,
• Each edge is an “intersection set,” naming the variables that overlap between the two cliques.
2.d. Sum-Product

Suppose we need $P(B,G)$:

1. **Product:** $P(B,C,D,F) = P(B)P(C|B)P(D|B)P(F|D)$
2. **Sum:** $P(B, C, F) = \sum_D P(B, C, D, F')$
3. **Product:** $P(B,C,E,F)=P(B,C,F)P(E|C)$
4. **Sum:** $P(B, E, F) = \sum_C P(B, C, E, F')$
5. **Product:** $P(B,E,F,G) = P(B,E,F)P(G|E,F)$
6. **Sum:** $P(B, G) = \sum_E \sum_F P(B, E, F, G)$

Complexity: $O\{NK^M\}$, where $N=\#$ cliques,
$K = \#$ values for each variable,
$M = 1 + \#$ variables in the largest clique
Consider the burglar alarm example.
a. Moralize this graph
b. Is it already triangulated? If not, triangulate it.
c. Draw the junction tree
Solution

a. Moralize this graph
Solution

b. Is it already triangulated?

Answer: yes. There is no unbroken cycle of length > 3.
Solution

c. Draw the junction tree

![Diagram with nodes ABE, AJ, and AM connected by edges labeled A]
Time Complexity of Bayes Net Inference

- Tree-structured Bayes nets: the sum-product algorithm
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  - The SAT problem is a Bayes net!
- Parameter Learning
Bayesian network inference

• In full generality, NP-hard
  • More precisely, #P-hard: equivalent to counting satisfying assignments

• We can reduce satisfiability to Bayesian network inference
  • Decision problem: is \( P(Y) > 0 \)?

\[
Y = (U_1 \lor U_2 \lor U_3) \land (\neg U_1 \lor \neg U_2 \lor U_3) \land (U_2 \lor \neg U_3 \lor U_4)
\]
Bayesian network inference

• In full generality, NP-hard
  • More precisely, \#P-hard: equivalent to counting satisfying assignments

• We can reduce *satisfiability* to Bayesian network inference
  • Decision problem: is \( P(Y) > 0 \)?

\[
Y = (U_1 \lor U_2 \lor U_3) \land (\neg U_1 \lor \neg U_2 \lor U_3) \land (U_2 \lor \neg U_3 \lor U_4)
\]

G. Cooper, 1990
Bayesian network inference

\[ P(U_1, U_2, U_3, U_4, C_1, C_2, C_3, D_1, D_2, Y) = \]

\[ P(U_1)P(U_2)P(U_3)P(U_4) \]
\[ P(C_1 | U_1, U_2, U_3)P(C_2 | U_1, U_2, U_3)P(C_3 | U_2, U_3, U_4) \]
\[ P(D_1 | C_1)P(D_2 | D_1, C_2)P(Y | D_2, C_3) \]
Bayesian network inference

Why can’t we use the junction tree algorithm to efficiently compute $\Pr(Y)$?
Why can’t we use the junction tree algorithm to efficiently compute $\Pr(Y)$?
Answer: after we moralize and triangulate, the size of the largest clique $(u_2 u_3 c_1 c_2 c_3)$ is $M \approx N$, same order of magnitude as the original problem.
Time Complexity of Bayes Net Inference

• Tree-structured Bayes nets: the sum-product algorithm
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• Parameter Learning
Parameter learning

- **Inference problem**: given values of evidence variables $E = e$, answer questions about query variables $X$ using the posterior $P(X | E = e)$

- **Learning problem**: estimate the parameters of the probabilistic model $P(X | E)$ given a *training sample* $\{(x_1,e_1), \ldots, (x_n,e_n)\}$
Parameter learning: complete data

• Suppose we know the network structure (but not the parameters), and have a training set of complete observations.
Parameter learning

• Suppose we know the network structure (but not the parameters), and have a training set of complete observations

• Example:

\[
P(S = T | C = T) = \frac{\# \text{ samples with } S = T, C = T}{\# \text{ samples with } C = T} = \frac{1}{4}
\]

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Parameter learning

• Suppose we know the network structure (but not the parameters), and have a training set of complete observations
  • $P(X \mid \text{Parents}(X))$ is given by the observed frequencies of the different values of $X$ for each combination of parent values
Parameter learning: missing data

- Suppose we know the network structure (but not the parameters), and have a training set, but the training set is *missing some observations.*
Missing data: the EM algorithm

- The EM algorithm starts ("Expectation Maximization") starts with an initial guess for each parameter value.
- We try to improve the initial guess, using the algorithm on the next two slides:
  - E-step
  - M-step

### Training set

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Missing data: the EM algorithm

• E-Step (Expectation): Given the model parameters, replace each of the missing numbers with a probability (a number between 0 and 1) using

\[
P(C = 1|S, R, W) = \frac{P(C = 1, S, R, W)}{P(C = 1, S, R, W) + P(C = 0, S, R, W)}
\]
Missing data: the EM algorithm

- M-Step (Maximization): Given the missing data estimates, replace each of the missing model parameters using

\[
P(\text{Variable} = T|\text{Parents} = \text{value}) = \frac{E[\# \text{ times Variable} = T, \text{Parents} = \text{value}]}{E[\#\text{times Parents} = \text{value}]}
\]

Training set

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Missing data: the EM algorithm

- Iterate back and forth between E-step and M-step until the model converges.

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Summary: Bayesian networks

- Structure
- Parameters
- Inference
- Learning