



Probability Refresher and Cycle Analysis

[A Quick Probability Refresher]

- A random variable, X , can take on a number of different possible values
 - Example: the number of pigeons on the windowsill outside is a random variable with possible values $1, 2, 3, \dots$
- Each time we observe (or sample) the random variable, it may take on a different value



[A Quick Probability Refresher]

- A random variable takes on each of these values with a specified probability
 - Example: $X = \{0, 1, 2, 3, 4\}$
 - $P[X=0] = .1, P[X=1] = .2, P[X=2] = .4, P[X=3] = .1, P[X=4] = .2$
- The sum of the probabilities of all values equals 1
 - $\sum_{\text{all values}} P[X=\text{value}] = 1$



A Quick Probability Refresher

■ Example

- Suppose we throw two dice and the random variable, X , is the sum of the two dice
- Possible values of X are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
- $P[X=2] = P[X=12] = 1/36$
- $P[X=3] = P[X=11] = 2/36$
- $P[X=4] = P[X=10] = 3/36$
- $P[X=5] = P[X=9] = 4/36$
- $P[X=6] = P[X=8] = 5/36$
- $P[X=7] = 6/36$

Note: $\sum_{i=2}^{12} P[X=i] = 1$



[A Quick Probability Refresher]

■ Expected Value

- Can be thought of a “long term average” of observing the random variable a large number of times

$$E[X] = \bar{x} = \sum_{\text{All possible values of } x} \text{Value} * P[X = \text{value}]$$

■ Example: dice - $E[X]$

$$= 2*1/36 + 3*2/36 + 4*3/36 + 5*4/36 + 6*5/36 + 7*6/36 + 8*5/36 + 9*4/36 + 10*3/36 + 11*2/36 + 12*1/36$$



[Probability Example]

- Basic probability notions
 - Two useful rules
 - Probabilities of all possible events sum to 1
 - Probability of independent events
 - Product of probabilities of events
 - e.g., probability of two coins coming up heads
 $= 1/2 \times 1/2 = 1/4$
 - Calculating averages/expected values
 - Function f
 - Multiply f by probability for each possible event
 - Sum over all events



Probability Example - Problem

- Given a bag with N balls
 - 1 *blue* ball
 - $N - 1$ *white* balls
- Algorithm
 - pick a ball
 - if *blue*, you win
 - else return to bag
 - repeat N times
- Question
 - What is your chance of winning for large N ?



Probability Example - Solution

- Can write as a sum
 - Chance of finding *blue* on first try = $1/N$
 - On second try = $[(N-1)/N] * (1/N)$
 - Etc.
- Instead, write
 - $1 - (\text{chance of losing})$
 - Parenthesized term
 - Product of N factors
 - Each factor = $(N-1)/N$
 - $1 - [(N - 1)/N]^N$



Probability Example - Solution

- For $N = 2$,
 - $1/2$ first is *white*
 - $1/2$ second is *white*
 - $1/4$ both are *white*
 - $3/4$ chance to win = $1 - (1/2)^2$
- For $N=3$,
 - $2/3$ first is *white*
 - $2/3$ second is *white*
 - $2/3$ third is *white*
 - $8/27$ all three are *white*
 - $19/27$ chance to win = $1 - (2/3)^3$ ($< 3/4$)



[Probability Example - Solution]

- $N=4$ probability of win = 68%
- $N=5$ probability of win = 67%
- $N=8$ probability of win = 66%
- large N ? 0 ?

$$\lim_{N \rightarrow \infty} \left(\frac{N-1}{N} \right)^N$$



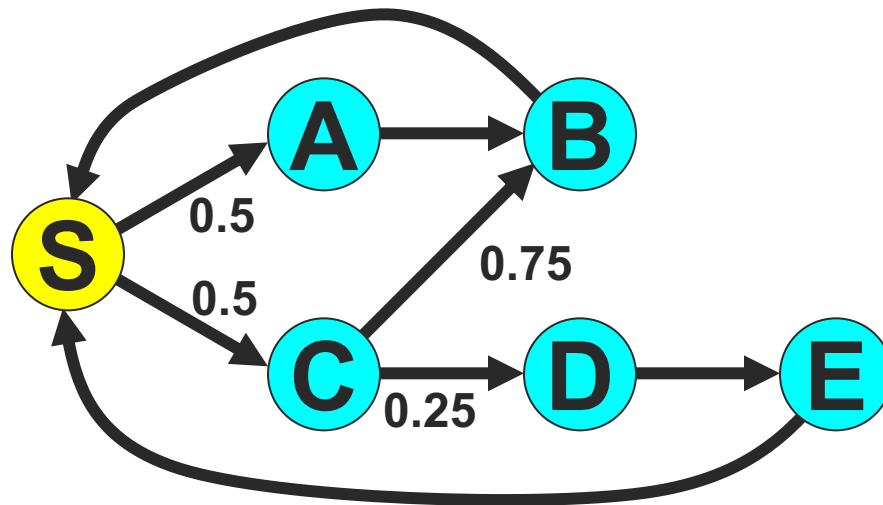
[Fun Example]

- Flip a coin repeatedly.
 - Two heads in a row scores 1 point.
 - Scoring pairs may not overlap
 - (e.g., three heads in a row does not score 2 points).
- On average, how many points do you score per flip?



[A Different Example]

- What fraction of time (on average) is spent in state E?



[Cycle Analysis]

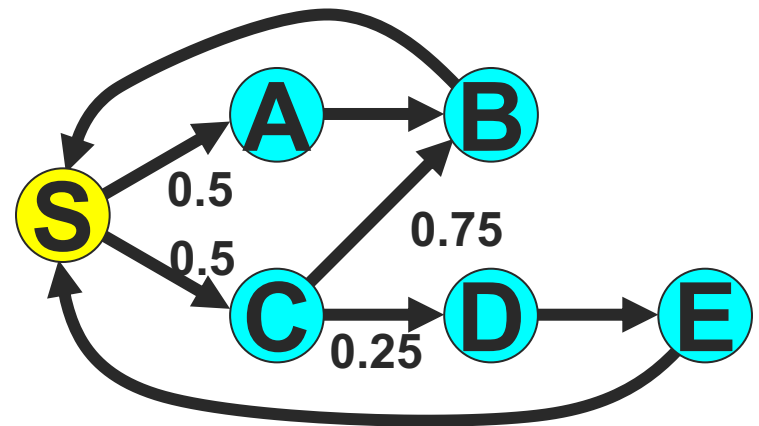
- Start with a discrete Markov process
 - Transitions happen periodically (every Δt)
 - Probabilities independent of past/future behavior
- Form all possible cyclic sequences (cycles)
 - Pick a “start” state
 - List all cycles from that state
 - Calculate probability per cycle
 - Calculate average cycle length
- Can calculate expected values of cycle-dependent properties with average length and cycle probabilities



[Example]

cycle

probability



Example

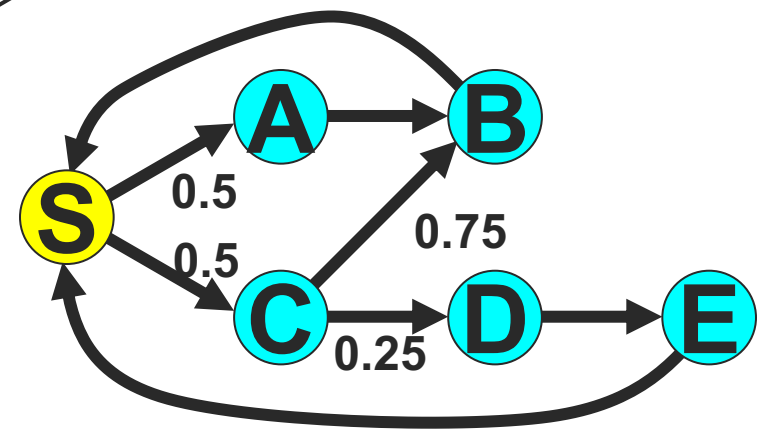
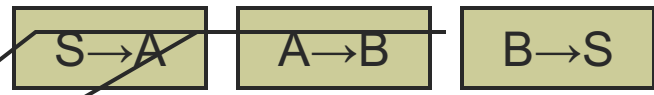
cycle

ABS

CBS

CDES

probability



average cycle length



Example

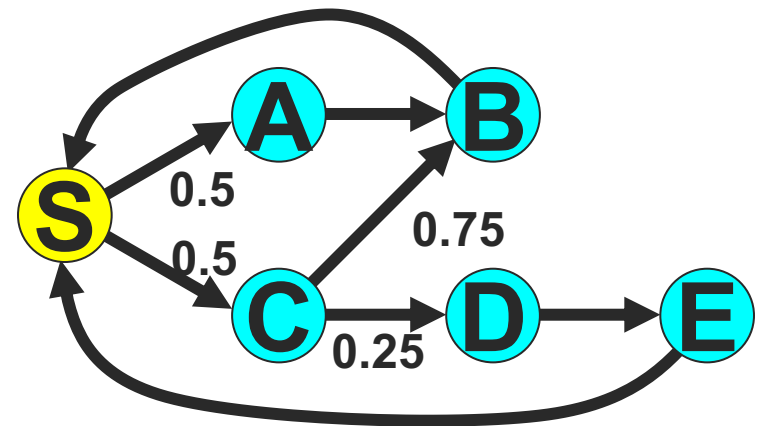
- average fraction of time spent in E

$$= 1 \cdot 0.125 \text{ periods/cycle}$$

- dividing by average length...

$$= 0.125 / 3.125 = 0.04$$

Amount of time spent in E when in cycle CDES



Probability of cycle CDES



[Fun Example]

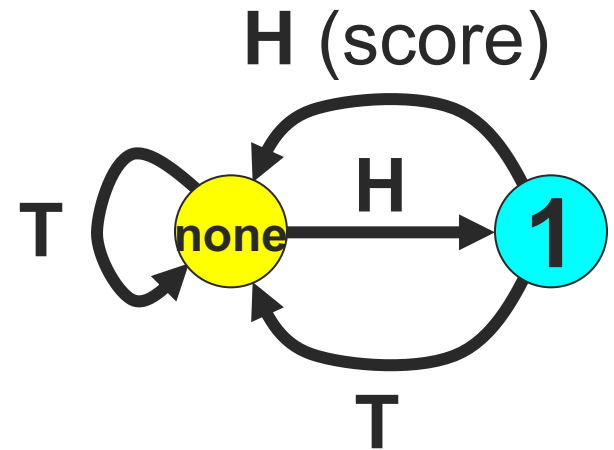
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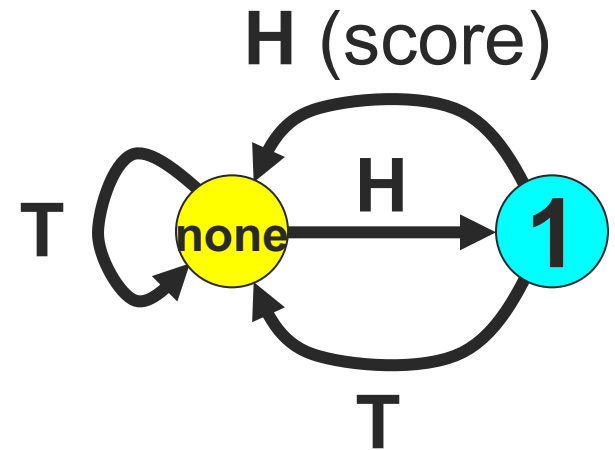
- cycle probability
- T 1/2
- HT 1/4
- HH 1/4

average cycle length
average score per cycle
average score per flip



[Fun Example]

- cycle probability
- T 1/2
- HT 1/4
- HH 1/4



average cycle length

$$= 1/2 + 1/2 + 1/2 = 3/2 \text{ flips}$$

average score per cycle

$$= 1/4 \text{ points}$$

average score per flip

$$= (1/4) / (3/2) = 1/6 \text{ pts/flip}$$

