Probability Refresher and Cycle Analysis
A random variable, $X$, can take on a number of different possible values

- Example: the number of pigeons on the windowsill outside is a random variable with possible values 1, 2, 3, ...

Each time we observe (or sample) the random variable, it may take on a different value
A Quick Probability Refresher

- A random variable takes on each of these values with a specified probability
  - Example: $X = \{0, 1, 2, 3, 4\}$
    - $P[X=0] = .1$, $P[X=1] = .2$, $P[X=2] = .4$, $P[X=3] = .1$, $P[X=4] = .2$
  - The sum of the probabilities of all values equals 1
    - $\sum_{\text{all values}} P[X=\text{value}] = 1$
Example
- Suppose we throw two dice and the random variable, $X$, is the sum of the two dice
- Possible values of $X$ are \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}
  - $P[X=2] = P[X=12] = 1/36$
  - $P[X=3] = P[X=11] = 2/36$
  - $P[X=5] = P[X=9] = 4/36$
  - $P[X=6] = P[X=8] = 5/36$
  - $P[X=7] = 6/36$

Note: $\sum_{i=2}^{12} P[X=i] = 1$
A Quick Probability Refresher

- **Expected Value**
  - Can be thought of a “long term average” of observing the random variable a large number of times

\[
E[X] = \bar{x} = \sum \text{Value} \times P[X = \text{value}]
\]

All possible values of \( x \)

- **Example: dice** - \( E[X] \)

\[
E[X] = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36}
\]
Probability Example

- Basic probability notions
  - Two useful rules
    - Probabilities of all possible events sum to 1
    - Probability of independent events
      - Product of probabilities of events
      - e.g., probability of two coins coming up heads
        \[ = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \]
  - Calculating averages/expected values
    - Function \( f \)
    - Multiply \( f \) by probability for each possible event
    - Sum over all events
Given a bag with \( N \) balls
- 1 blue ball
- \( N - 1 \) white balls

Algorithm
- pick a ball
  - if blue, you win
  - else return to bag
- repeat \( N \) times

Question
- What is your chance of winning for large \( N \)?
Probability Example - Solution

- Can write as a sum
  - Chance of finding *blue* on first try = $1/N$
  - On second try = $[(N-1)/N] \times (1/N)$
  - Etc.

- Instead, write
  - $1 - (\text{chance of losing})$
  - Parenthesized term
    - Product of $N$ factors
    - Each factor = $(N-1)/N$
  - $1 - [(N - 1)/N]^N$
Probability Example - Solution

- For $N = 2$,
  - $1/2$ first is white
  - $1/2$ second is white
  - $1/4$ both are white
  - $3/4$ chance to win = $1 - (1/2)^2$

- For $N=3$,
  - $2/3$ first is white
  - $2/3$ second is white
  - $2/3$ third is white
  - $8/27$ all three are white
  - $19/27$ chance to win = $1 - (2/3)^3$ ($< 3/4$)
Probability Example - Solution

- $N=4$ probability of win = 68%
- $N=5$ probability of win = 67%
- $N=8$ probability of win = 66%
- large $N$? $0$?

$$\lim_{N \to \infty} \left( \frac{N-1}{N} \right)^N$$
Fun Example

- Flip a coin repeatedly.
  - Two heads in a row scores 1 point.
  - Scoring pairs may not overlap
    - (e.g., three heads in a row does not score 2 points).

- On average, how many points do you score per flip?
A Different Example

- What fraction of time (on average) is spent in state E?
Cycle Analysis

- Start with a discrete Markov process
  - Transitions happen periodically (every $\Delta t$)
  - Probabilities independent of past/future behavior
- Form all possible cyclic sequences (cycles)
  - Pick a “start” state
  - List all cycles from that state
  - Calculate probability per cycle
  - Calculate average cycle length
- Can calculate expected values of cycle-dependent properties with average length and cycle probabilities
Example

cycle  probability

- Cycle A
- Cycle B
- Cycle C
- Cycle D
- Cycle E

- Probability P(A,S) = 0.5
- Probability P(B,A) = 0.5
- Probability P(B,C) = 0.75
- Probability P(C,D) = 0.25
- Probability P(D,E) = 0.25
- Probability P(E,B) = 0.75
Example

- **ABS**: Probability $= 0.5 \times 1 \times 1 = 0.5$
- **CBS**: Probability $= 0.5 \times 0.75 \times 1 = 0.375$
- **CDES**: Probability $= 0.5 \times 0.25 \times 1 \times 1 = 0.125$

**Average cycle length**

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<th>ABS</th>
<th>CBS</th>
<th>CDES</th>
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<tbody>
<tr>
<td></td>
<td>3.75</td>
<td>3.125</td>
<td>3.125</td>
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</table>
Example

- average fraction of time spent in E
  - $= 1 \cdot 0.125 \text{ periods/cycle}$
- dividing by average length...
  - $= 0.125 / 3.125 = 0.04$
Fun Example

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  - Two heads in a row scores 1 point.
  - Scoring pairs may not overlap
    - (e.g., three heads in a row does not score 2 points).
- On average, how many points do you score per flip?
Fun Example

- cycle probability
  - T: 1/2
  - HT: 1/4
  - HH: 1/4

average cycle length
average score per cycle
average score per flip
Fun Example

- cycle probability
  - T: 1/2
  - HT: 1/4
  - HH: 1/4

average cycle length: $= 1/2 + 1/2 + 1/2 = 3/2$ flips
average score per cycle: $= 1/4$ points
average score per flip: $= (1/4) / (3/2) = 1/6$ pts/flip