Probability Refresher and Cycle Analysis
A Quick Probability Refresher

- A random variable, \( X \), can take on a number of different possible values
  - Example: the number of pigeons on the windowsill outside is a random variable with possible values 1, 2, 3, ...

- Each time we observe (or sample) the random variable, it may take on a different value
A Quick Probability Refresher

- A random variable takes on each of these values with a specified probability
  - Example: $X = \{0, 1, 2, 3, 4\}$
  - $P[X=0] = .1, P[X=1] = .2, P[X=2] = .4, P[X=3] = .1, P[X=4] = .2$

- The sum of the probabilities of all values equals 1
  - $\sum_{all\ values} P[X=value] = 1$
A Quick Probability Refresher

Example
- Suppose we throw two dice and the random variable, $X$, is the sum of the two dice
- Possible values of $X$ are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
- $P[X=2] = P[X=12] = 1/36$
- $P[X=3] = P[X=11] = 2/36$
- $P[X=5] = P[X=9] = 4/36$
- $P[X=6] = P[X=8] = 5/36$
- $P[X=7] = 6/36$

Note: $\sum_{i=2}^{12} P[X=i] = 1$
A Quick Probability Refresher

- **Expected Value**
  - Can be thought of a “long term average” of observing the random variable a large number of times

\[
E[X] = \mu = \sum_{\text{All possible values of } x} \text{Value} \times P[X = \text{value}]
\]

- **Example: dice** -  
  \[
  E[X] = 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36} + 6\times\frac{5}{36} + 7\times\frac{6}{36} + 8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36}
  \]
Probability Example

- Basic probability notions
  - Two useful rules
    - Probabilities of all possible events sum to 1
    - Probability of independent events
      - Product of probabilities of events
      - e.g., probability of two coins coming up heads
        \[ \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \]
  - Calculating averages/expected values
    - Function \( f \)
    - Multiply \( f \) by probability for each possible event
    - Sum over all events
Probability Example - Problem

- Given a bag with $N$ balls
  - 1 blue ball
  - $N - 1$ white balls

- Algorithm
  - pick a ball
    - if blue, you win
    - else return to bag
  - repeat $N$ times

- Question
  - What is your chance of winning for large $N$?
Can write as a sum
- Chance of finding *blue* on first try = $1/N$
- On second try = $[(N-1)/N] \times (1/N)$
- Etc.

Instead, write
- $1 - (\text{chance of losing})$
- Parenthesized term
  - Product of $N$ factors
  - Each factor = $(N-1)/N$
- $1 - [(N - 1)/N]^N$
Probability Example - Solution

- For $N = 2$,
  - $1/2$ first is white
  - $1/2$ second is white
  - $1/4$ both are white
  - $3/4$ chance to win $= 1 - (1/2)^2$

- For $N=3$,
  - $2/3$ first is white
  - $2/3$ second is white
  - $2/3$ third is white
  - $8/27$ all three are white
  - $19/27$ chance to win $= 1 - (2/3)^3$ ($< 3/4$)
Probability Example - Solution

- $N=4$ probability of win = 68%
- $N=5$ probability of win = 67%
- $N=8$ probability of win = 66%
- large $N$? 0?

$$\lim_{N \to \infty} \left( \frac{N - 1}{N} \right)^N$$
Fun Example

- Flip a coin repeatedly.
  - Two heads in a row scores 1 point.
  - Scoring pairs may not overlap
    - (e.g., three heads in a row does not score 2 points).

- On average, how many points do you score per flip?
A Different Example

What fraction of time (on average) is spent in state E?
Cycle Analysis

- Start with a discrete Markov process
  - Transitions happen periodically (every $\Delta t$)
  - Probabilities independent of past/future behavior

- Form all possible cyclic sequences (cycles)
  - Pick a “start” state
  - List all cycles from that state
  - Calculate probability per cycle
  - Calculate average cycle length

- Can calculate expected values of cycle-dependent properties with average length and cycle probabilities
Example

cycle probability

![Diagram with nodes A, B, C, D, E connected with probabilities 0.5, 0.75, 0.25]
Example

- **cycle probability**
  - **ABS**
  - **CBS**
  - **CDES**

**average cycle length**

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Example

- average fraction of time spent in E
  
  \[ \text{time spent in E} = 1 \cdot 0.125 \text{ periods/cycle} \]

- dividing by average length…
  
  \[ \frac{0.125}{3.125} = 0.04 \]
Fun Example

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    - (e.g., three heads in a row does not score 2 points).

- On average, how many points do you score per flip?
Fun Example

- cycle probability
  - T: 1/2
  - HT: 1/4
  - HH: 1/4

average cycle length
average score per cycle
average score per flip
Fun Example

- cycle probability
- T 1/2
- HT 1/4
- HH 1/4

average cycle length = 1/2 + 1/2 + 1/2 = 3/2 flips
average score per cycle = 1/4 points
average score per flip = (1/4) / (3/2) = 1/6 pts/flip