Probability Refresher and Cycle Analysis
A Quick Probability Refresher

- A random variable, $X$, can take on a number of different possible values
  - Example: the number of pigeons on the windowsill outside is a random variable with possible values 1,2,3,…

- Each time we observe (or sample) the random variable, it may take on a different value
A Quick Probability Refresher

- A random variable takes on each of these values with a specified probability
  - Example: \( X = \{0, 1, 2, 3, 4\} \)
  - \( P[X=0] = .1, P[X=1] = .2, P[X=2] = .4, P[X=3] = .1, P[X=4] = .2 \)

- The sum of the probabilities of all values equals 1
  - \( \sum_{all \ values} P[X=value] = 1 \)
A Quick Probability Refresher

Example

- Suppose we throw two dice and the random variable, $X$, is the sum of the two dice
- Possible values of $X$ are \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}
- $P[X=2] = P[X=12] = 1/36$
- $P[X=3] = P[X=11] = 2/36$
- $P[X=5] = P[X=9] = 4/36$
- $P[X=6] = P[X=8] = 5/36$
- $P[X=7] = 6/36$

Note: $\sum_{i=2}^{12} P[X=i] = 1$
A Quick Probability Refresher

A probability distribution function matches each possible value of a random variable with its associated probability.
A Quick Probability Refresher

- The cumulative distribution function of a random variable, $X$, is defined by
  - CDF: $P[X \leq x] = \sum_{\text{all } y=x} P[x=y]$
A Quick Probability Refresher

- **Expected Value**
  - Can be thought of a “long term average” of observing the random variable a large number of times

\[
E[X] = \overline{x} = \sum \text{Value} \times P[X = \text{value}]
\]

- **Example: dice** - \( E[X] \)
A Quick Probability Refresher

- **Average vs. Expected Value**
  - Short term average
    - Suppose a random variable $X$ is sampled $N$ times
    - Let $n_i = \# \text{ of } X = i \text{ was observed}$
    - Average of samples
      - $= \frac{n_0 \times 0 + n_1 \times 1 + n_2 \times 2 + n_3 \times 3 + ...}{N}$
      - $= \frac{n_0}{N \times 0} + \frac{n_1}{N \times 1} + \frac{n_2}{N \times 2} + \frac{n_3}{N \times 3} + ...$
    - As $N \to \infty$, the ratio $n_i/N$ becomes $p_i$
  - Thus, $E[X]$
    - $= \lim_{N \to \infty} \left[ \frac{n_0}{N \times 0} + \frac{n_1}{N \times 1} + \frac{n_2}{N \times 2} + \frac{n_3}{N \times 3} + ... \right]$
A Quick Probability Refresher

Continuous Random Variables

- In many cases, a random variable takes a value drawn from a continuous interval
  - Ex: processing time for a packet may be any real value \([0, \infty)\)
- The distribution of possible values a continuous random variable can take is given by a probability density function, \(F(x)\)
- \(P(a \leq x \leq b) = \int_a^b F(x)dx = \sum_{i=a}^{b} P(x = i)\)
- \(E[x] = \int_{-\infty}^{\infty} xf(x)dx = \sum_i * P(x = i)\)
Probability Example

- Basic probability notions
  - Two useful rules
    - Probabilities of all possible events sum to 1
    - Probability of independent events
      - Product of probabilities of events
      - e.g., probability of two coins coming up heads
        $$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
  - Calculating averages/expected values
    - Function $f$
    - Multiply $f$ by probability for each possible event
    - Sum over all events
Probability Example - Problem

- Given a bag with \( N \) balls
  - 1 blue ball
  - \( N - 1 \) white balls

- Algorithm
  - pick a ball
    - if blue, you win
    - else return to bag
  - repeat \( N \) times

- Question
  - What is your chance of winning for large \( N \)?
Probability Example - Solution

- Can write as a sum
  - Chance of finding blue on first try = 1/N
  - On second try = \((N-1)/N\) * (1/N)
  - Etc.

- Instead, write
  - 1 - (chance of losing)
  - Parenthesized term
    - Product of \(N\) factors
    - Each factor = \((N-1)/N\)
  - 1 - \([(N - 1)/N]^N\)
Probability Example - Solution

- For \( N = 2 \),
  - 1/2 first is \textit{white}
  - 1/2 second is \textit{white}
  - 1/4 both are \textit{white}
  - 3/4 chance to win = 1 - (1/2)^2

- For \( N = 3 \),
  - 2/3 first is \textit{white}
  - 2/3 second is \textit{white}
  - 2/3 third is \textit{white}
  - 8/27 all three are \textit{white}
  - 19/27 chance to win = 1 - (2/3)^3 \ (< 3/4)
Probability Example - Solution

- \(N=4\) probability of win = 68%
- \(N=5\) probability of win = 67%
- \(N=8\) probability of win = 66%
- large \(N\)? \(0\)?

\[
\lim_{N \to \infty} \left( \frac{N - 1}{N} \right)^N
\]
Fun Example

- Flip a coin repeatedly.
  - Two heads in a row scores 1 point.
  - Scoring pairs may not overlap
    - (e.g., three heads in a row does not score 2 points).

- On average, how many points do you score per flip?

- Would you play this game in Las Vegas for
  - $1 per flip and $5 per point?
  - $1 per flip and $7 per point?
A Different Example

What fraction of time (on average) is spent in state E?
Cycle Analysis

- Start with a discrete Markov process
  - Transitions happen periodically (every $\Delta t$)
  - Probabilities independent of past/future behavior
- Form all possible cyclic sequences (cycles)
  - Pick a “start” state
  - List all cycles from that state
  - Calculate probability per cycle
  - Calculate average cycle length
- Can calculate expected values of cycle-dependent properties with average length and cycle probabilities
Example

cycle

probability

- \text{S}
- A
- B
- C
- D
- E

\begin{align*}
\text{probability} & = \begin{bmatrix}
0.5 & 0.5 & 0.25 & 0.75 & 0.25 \\
0.5 & 0.75 & 0.5 & 0.25 & 0.75 \\
0.25 & 0.5 & 0.75 & 0.5 & 0.5 \\
0.75 & 0.5 & 0.25 & 0.5 & 0.75 \\
0.25 & 0.25 & 0.75 & 0.5 & 0.5 \\
\end{bmatrix}
\end{align*}
Example

**Example**

<table>
<thead>
<tr>
<th>Cycle</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABS</td>
<td>0.5</td>
</tr>
<tr>
<td>CBS</td>
<td>0.375</td>
</tr>
<tr>
<td>CDES</td>
<td>0.125</td>
</tr>
</tbody>
</table>

**Average Cycle Length**

\[
\text{Average Cycle Length} = 3 \times 0.5 + 3 \times 0.375 + 4 \times 0.125 = 3.125
\]
Example

- average fraction of time spent in E
  - $1 \times 0.125$ periods/cycle
  - dividing by average length...
  - $0.125 / 3.125 = 0.04$
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Fun Example

- cycle probability
- T 1/2
- HT 1/4
- HH 1/4

average cycle length
average score per cycle
average score per flip

Diagram:

- Initial state: T
- Possible transitions:
  - T → H
  - H → T

Final state: 1

H (score)
Fun Example

- cycle probability
- T 1/2
- HT 1/4
- HH 1/4

average cycle length = 1/2 + 1/2 + 1/2 = 3/2 flips
average score per cycle = 1/4 points
average score per flip = (1/4) / (3/2) = 1/6 pts/flip

(Good luck getting $7 per point in Vegas!)