CS 421 Lecture 22: Operational semantics

- Announcements

- Lecture outline
  - Operational semantics of OCaml
  - OS_{subst}: substitution model
  - OS_{clo}: closure model
  - OS_{state}: closure model + state

Announcements

- MP8 has been posted
  - Higher-order functions
  - Last (real) MP

- Reminder: 4-unit grad projects
  - Status report due tomorrow

Operational Semantics of OCaml

- Present three systems
  - OS_{subst} : substitution model
  - OS_{clo} : closure model
  - OS_{state} : closure model, plus state
- In all systems, start with removing let’s and letrec’s as follows:
  - let x = e in e′ => (fun x -> e′) e
  - let rec f = e in e′ => (fun f -> e′) (rec f = e)
- Here, “rec f = e” is a new expression added just for the operational semantics.
  - Note: e must be an abstraction.

OS_{subst}

- Just like OSsimp, but with recursion.
- Expressions:
  - consts (not higher order), vars, application, abstraction, built-in function calls: $e_1 \oplus e_2$
  - Note: Partial application of built-ins is not allowed. This implies that in an application e_1 e_2, e_1 must reduce to a user-defined function.
- Values:
  - consts, (closed) abstractions
- Judgments:
  - $e \Downarrow v$ (e closed)

Axioms

- (Const)

$$\overline{k \Downarrow k}$$

- (Abstr)

$$\overline{\text{fun } x \rightarrow e \Downarrow \text{fun } x \rightarrow e}$$

- (Rec)

$$\overline{\text{rec } f = e \Downarrow e[\text{rec } f = e / f]}$$

Remember: e is an abstraction

Rules of infrerence (same as Os_{simp})

- (δ rules) $$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v = v_1 \oplus v_2}{e_1 \oplus e_2 \Downarrow v}$$

- (if-true) $$\frac{e_1 \Downarrow \mathit{true} \qquad e_2 \Downarrow v}{\text{if } e_1 \text{ then } e_2 \text{ else } e_3 \Downarrow v}$$

- (if-false) $$\frac{e_1 \Downarrow \mathit{false} \qquad e_3 \Downarrow v}{\text{if } e_1 \text{ then } e_2 \text{ else } e_3 \Downarrow v}$$

Never gave these in OS_{simp}

Rules of infrerence (cont.)

- (application)

$$\frac{e_1 \Downarrow \text{fun } x \rightarrow e \quad e_2 \Downarrow v \quad e[v/x] \Downarrow v'}{e_1 e_2 \Downarrow v'}$$

Example

let rec f = fun n -> if n=1 then 1 else n*f(n-1) in f(2) $\Downarrow$ 2

OS_{clo}

- Same expressions; same translation of let and letrec.
- Definition:
  - Environments (notated η): map from variable to value
  - Closures = Expression $\times$ Env (notated $<e, \eta>$)

must be an abstraction

must contain a value for every free variable in expression

  - Note that there is a circularity here: Env's contain closures and closures contain env's. We'll just keep this informal.

Axioms of OS_{clo}

- (Const) $\overline{\eta, k \Downarrow k}$ (Var) $\overline{\eta, x \Downarrow \eta(x)}$
- (Abstr) $\overline{\eta, \text{fun } x \to e \Downarrow \langle \text{fun } x \to e, \eta \rangle}$
- (Rec) $\overline{\eta, \text{rec } f = e \Downarrow \langle e, \eta' \rangle}$

  where $\eta' = \eta[f \to \langle e, \eta' \rangle]$

Again, be informal about this.

Also, again note that e is an abstraction.

Rules of infrerence

- (δ) $$\frac{\eta, e_1 \Downarrow v_1 \quad \eta, e_2 \Downarrow v_2 \quad v = v_1 \oplus v_2}{\eta, e_1 \oplus e_2 \Downarrow v}$$

- (App)

$$\frac{\eta, e_1 \Downarrow \langle \text{fun } x \rightarrow e, \eta' \rangle \quad \eta, e_2 \Downarrow v \quad \eta'[x \rightarrow v] \Downarrow v'}{\eta, e_1 e_2 \Downarrow v'}$$

Example

$$\frac{}{\varnothing, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3)4 \Downarrow 7}$$

Example

$$\dfrac{\overline{\varnothing, (\text{fun x} \rightarrow (\text{fun y} \rightarrow \text{x+y})3) \Downarrow \text{<fun x} \rightarrow (\text{fun y} \rightarrow \text{x+y})3, \varnothing\text{>}} \quad \overline{\varnothing, 4 \Downarrow 4} \quad \mathbf{A}}{\varnothing, (\text{fun x} \rightarrow (\text{fun y} \rightarrow \text{x+y})3)4 \Downarrow 7}$$

Example

$$\overline{\mathbf{A} = [x{:}4],\ (\text{fun } y \rightarrow x{+}y)3 \Downarrow 7}$$

$$\frac{\overline{\varnothing,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3) \Downarrow \langle \text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3, \varnothing \rangle} \qquad \overline{\varnothing,\ 4 \Downarrow 4} \qquad \mathbf{A}}{\varnothing,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3)4 \Downarrow 7}$$

Example

$$\frac{\overline{[x:4],\ (\text{fun } y \rightarrow x+y) \Downarrow \langle \text{fun } y \rightarrow x+y\ ,[x:4]\rangle} \qquad \overline{[x:4],\ 3 \Downarrow 3} \qquad \textbf{B}}{\textbf{A} = [x:4],\ (\text{fun } y \rightarrow x+y)3 \Downarrow 7}$$

$$\frac{\overline{\varnothing,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3) \Downarrow \langle \text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3, \varnothing\rangle} \qquad \overline{\varnothing,\ 4 \Downarrow 4} \qquad \textbf{A}}{\varnothing,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3)4 \Downarrow 7}$$

Example

$$\frac{\overline{[x:4,y:3],\ x \Downarrow 4} \qquad \overline{[x:4,y:3],\ y \Downarrow 3}}{\mathbf{B =}\ [x:4,y:3],\ x+y \Downarrow 7}$$

$$\frac{\overline{[x:4],\ (\text{fun } y \to x+y) \Downarrow \langle \text{fun } y \to x+y\ ,[x:4]\rangle} \qquad \overline{[x:4],\ 3 \Downarrow 3} \qquad \mathbf{B}}{\mathbf{A =}\ [x:4],\ (\text{fun } y \to x+y)3 \Downarrow 7}$$

$$\frac{\overline{\varnothing,\ (\text{fun } x \to (\text{fun } y \to x+y)3) \Downarrow \langle \text{fun } x \to (\text{fun } y \to x+y)3, \varnothing\rangle} \qquad \overline{\varnothing,\ 4 \Downarrow 4} \qquad \mathbf{A}}{\varnothing,\ (\text{fun } x \to (\text{fun } y \to x+y)3)4 \Downarrow 7}$$

OS_{state}

- Add:
  - Locations
    - Notated ℓ, ℓ', ℓ_1, *etc.*
    - Infinite, unstructured set of atoms
  - State
    - Notated σ
    - Map from locations to values
  - Values: Constants, locations, closures
  - Judgments:

$$\sigma,\eta \vdash e \Downarrow v,\sigma'$$

Axioms and rules of inference

- All rules are same as OS_{clo}, but "thread" state through subcomputation. States are never captured in closures.
- (Const) $\overline{\sigma, \eta \vdash k \Downarrow k, \sigma}$ (Var) $\overline{\sigma, \eta \vdash x \Downarrow \eta(x), \sigma}$
- (Abstr) $$\overline{\sigma, \eta \vdash \text{fun } x \rightarrow e \Downarrow \langle \text{fun } x \rightarrow e, \eta \rangle, \sigma}$$
- (δ) $$\frac{\sigma, \eta \vdash e_1 \Downarrow v_1, \sigma_1 \quad \sigma_1, \eta \vdash e_2 \Downarrow v_2, \sigma_2 \quad v = v_1 \oplus v_2}{\sigma, \eta \vdash e_1 \oplus e_2 \Downarrow v, \sigma_2}$$

Axioms and rules of inference

- New rules for operators

- (Deref)

$$\frac{\sigma,\eta \vdash e \Downarrow \ell,\sigma' \qquad \ell \text{ a location} \qquad \sigma'=v}{\sigma,\eta \vdash\ !e \Downarrow v,\sigma'}$$

- (Assign)

$$\frac{\sigma,\eta \vdash e_1 \Downarrow \ell,\sigma' \qquad \sigma',\eta \vdash e_2 \Downarrow v,\sigma''}{\sigma,\eta \vdash e_1 := e_2 \Downarrow (),\sigma''[\ell \to v]}$$

unique value of type unit

- (Ref)

$$\frac{\sigma,\eta \vdash e \Downarrow v,\sigma' \qquad \ell \text{ a fresh location}}{\sigma,\eta \vdash \text{ref } e \Downarrow \ell,\sigma'[\ell \to v]}$$

Example

$$\frac{}{\{\ell:0\},\{x:\ell\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y{:=}!y+1))x \Downarrow 1,\{\ell:1\}}$$

Example

$$\dfrac{\dfrac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun y} \rightarrow \ldots) \Downarrow \langle(\text{fun y} \rightarrow \ldots),\ \{x{:}\ell\}\rangle, \{\ell{:}0\}} \quad \dfrac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash x \Downarrow \ell, \{\ell{:}0\}} \quad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun y} \rightarrow (\text{fun z} \rightarrow !x)(y{:=}!y{+}1))x \Downarrow 1, \{\ell{:}1\}}$$

Example

$$\dfrac{\overline{\{\ell:0\},\{x:\ell,y:\ell\} \vdash (\text{fun } z \rightarrow !x) \Downarrow \langle \text{fun } z \rightarrow !x, \{x:\ell,y:\ell\}\rangle, \{\ell:0\}} \qquad \mathbf{B} \qquad \mathbf{C}}{\mathrm{A} = \{\ell:0\},\{x:\ell,y:\ell\} \vdash (\text{fun } z \rightarrow !x)(y:=!y+1) \Downarrow 1, \{\ell:1\}}$$

$$\dfrac{\overline{\{\ell:0\},\{x:\ell\} \vdash (\text{fun } y \rightarrow \ldots) \Downarrow \langle (\text{fun } y \rightarrow \ldots), \{x:\ell\}\rangle, \{\ell:0\}} \qquad \overline{\{\ell:0\},\{x:\ell\} \vdash x \Downarrow \ell, \{\ell:0\}} \qquad \mathbf{A}}{\{\ell:0\},\{x:\ell\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y:=!y+1))x \Downarrow 1, \{\ell:1\}}$$

Example

$$\frac{}{\color{red}{B =}\ \{\ell{:}0\},\{x{:}\ell, y{:}\ell\} \vdash y{:=}!y+1 \Downarrow (), \{\ell{:}1\}}$$

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell, y{:}\ell\} \vdash (\text{fun } z \rightarrow !x) \Downarrow \langle \text{fun } z \rightarrow !x, \{x{:}\ell, y{:}\ell\}\rangle, \{\ell{:}0\}} \quad \color{red}{B} \quad \color{red}{C}}{\color{red}{A =}\ \{\ell{:}0\},\{x{:}\ell, y{:}\ell\} \vdash (\text{fun } z \rightarrow !x)(y{:=}!y+1) \Downarrow 1, \{\ell{:}1\}}$$

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun } y \rightarrow \ldots) \Downarrow \langle (\text{fun } y \rightarrow \ldots), \{x{:}\ell\}\rangle, \{\ell{:}0\}} \quad \frac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash x \Downarrow \ell, \{\ell{:}0\}} \quad \color{red}{A}}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y{:=}!y+1))x \Downarrow 1, \{\ell{:}1\}}$$

Example

$$\frac{\dfrac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y \Downarrow \ell,\{\ell{:}0\}} \qquad \dfrac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash\ !y+1 \Downarrow 1,\{\ell{:}0\}}}{\mathbf{B} = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y{:=}!y+1 \Downarrow (),\{\ell{:}1\}}$$

$$\frac{\dfrac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash (\mathrm{fun}\ z \rightarrow !x) \Downarrow <\mathrm{fun}\ z \rightarrow !x, \{x{:}\ell,y{:}\ell\}>,\{\ell{:}0\}} \qquad \mathbf{B} \qquad \mathbf{C}}{\mathrm{A} = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash (\mathrm{fun}\ z \rightarrow !x)(y{:=}!y+1) \Downarrow 1,\{\ell{:}1\}}$$

$$\frac{\dfrac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash (\mathrm{fun}\ y \rightarrow \ldots) \Downarrow <(\mathrm{fun}\ y \rightarrow \ldots),\ \{x{:}\ell\}>,\{\ell{:}0\}} \qquad \dfrac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash x \Downarrow \ell,\{\ell{:}0\}} \qquad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\}\vdash (\mathrm{fun}\ y \rightarrow (\mathrm{fun}\ z \rightarrow !x)(y{:=}!y+1))x \Downarrow 1,\{\ell{:}1\}}$$

Example

$$\dfrac{\dfrac{\dfrac{}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash y \Downarrow \ell,\{\ell:0\}} \qquad \{\ell:0\}(\ell)=0}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash\ !y \Downarrow 0,\{\ell:0\}} \qquad \dfrac{}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash 1 \Downarrow 1,\{\ell:0\}}}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash\ !y+1 \Downarrow 1,\{\ell:0\}}$$

$$\dfrac{\dfrac{}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash y \Downarrow \ell,\{\ell:0\}} \qquad \{\ell:0\},\{x:\ell,y:\ell\}\vdash\ !y+1 \Downarrow 1,\{\ell:0\}}{\mathbf{B} = \{\ell:0\},\{x:\ell,y:\ell\}\vdash y:=!y+1 \Downarrow (),\{\ell:1\}}$$

$$\dfrac{\dfrac{}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash (\text{fun } z \to !x) \Downarrow \langle \text{fun } z \to !x, \{x:\ell,y:\ell\}\rangle,\{\ell:0\}} \qquad \mathbf{B} \qquad \mathbf{C}}{\mathrm{A} = \{\ell:0\},\{x:\ell,y:\ell\}\vdash (\text{fun } z \to !x)(y:=!y+1) \Downarrow 1,\{\ell:1\}}$$

$$\dfrac{\dfrac{}{\{\ell:0\},\{x:\ell\}\vdash (\text{fun } y \to \ldots) \Downarrow \langle (\text{fun } y \to \ldots),\ \{x:\ell\}\rangle,\{\ell:0\}} \qquad \dfrac{}{\{\ell:0\},\{x:\ell\}\vdash x \Downarrow \ell,\{\ell:0\}} \qquad \mathbf{A}}{\{\ell:0\},\{x:\ell\}\vdash (\text{fun } y \to (\text{fun } z \to !x)(y:=!y+1))x \Downarrow 1,\{\ell:1\}}$$

Example

$$\frac{\overline{\{\ell:1\},\{x:\ell,y:\ell,z:()\}\vdash x \Downarrow \ell,\{\ell:1\}} \qquad \{\ell:1\}(\ell)=1}{\mathbf{C} = \{\ell:1\},\{x:\ell,y:\ell,z:()\}\vdash !x \Downarrow 1,\{\ell:1\}}$$