Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Sequencing

$$\frac{\{P\}\ C_1\ \{Q\} \qquad \{Q\}\ C_2\ \{R\}}{\{P\}\ C_1;\ C_2\ \{R\}}$$

- Example:

$$\frac{\begin{array}{c}\{z = z \ \&\ z = z\}\ x := z\ \{x = z \ \&\ z = z\} \\ \{x = z \ \&\ z = z\}\ y := z\ \{x = z \ \&\ y = z\}\end{array}}{\{z = z \ \&\ z = z\}\ x := z;\ y := z\ \{x = z \ \&\ y = z\}}$$

Sequencing

$$\frac{\{P\}\ C_1\ \{Q\} \qquad \{Q\}\ C_2\ \{R\}}{\{P\}\ C_1;\ C_2\ \{R\}}$$

- Example:

$$\frac{\{z = z \ \&\ z = z\}\ x := z\ \{x = z \ \&\ z = z\} \qquad \{x = z \ \&\ z = z\}\ y := z\ \{x = z \ \&\ y = z\}}{\{z = z \ \&\ z = z\}\ x := z;\ y := z\ \{x = z \ \&\ y = z\}}$$

Postcondition Weakening

$$\frac{\{P\}\ C\ \{Q'\} \quad Q' \rightarrow Q}{\{P\}\ C\ \{Q\}}$$

Example:

$$\frac{\{z = z\ \&\ z = z\}\ x := z;\ y := z\ \{x = z\ \&\ y = z\} \qquad (x = z\ \&\ y = z) \rightarrow (x = y)}{\{z = z\ \&\ z = z\}\ x := z;\ y := z\ \{x = y\}}$$

Rule of Consequence

$$\frac{P \rightarrow P' \qquad \{P'\}\ C\ \{Q'\} \qquad Q' \rightarrow Q}{\{P\}\ C\ \{Q\}}$$

- Logically equivalent to the combination of Precondition Strengthening and Postcondition Weakening
- Uses $P \rightarrow P'$ and $Q' \rightarrow Q$

If Then Else

$$\frac{\{P \text{ and } B\}\ C_1\ \{Q\} \quad \{P \text{ and } (\text{not } B)\}\ C_2\ \{Q\}}{\{P\}\ \textbf{if}\ B\ \textbf{then}\ C_1\ \textbf{else}\ C_2\ \textbf{fi}\ \{Q\}}$$

- Example: Want

{y=a}

if x < 0 then y:= y-x else y:= y+x fi

{y=a+|x|}

Suffices to show:

(1) {y=a&x<0} y:=y-x {y=a+|x|} and

(4) {y=a&not(x<0)} y:=y+x {y=a+|x|}

$\{y=a \& x<0\}\ \ y:=y-x\ \ \{y=a+|x|\}$

(3) $(y=a \& x<0) \rightarrow y-x=a+|x|$

(2) $\{y-x=a+|x|\}\ \ y:=y-x\ \ \{y=a+|x|\}$

(1) $\{y=a \& x<0\}\ \ y:=y-x\ \ \{y=a+|x|\}$

- (1) Reduces to (2) and (3) by Precondition Strengthening
- (2) Follows from assignment axiom
- (3) Because $x<0 \rightarrow |x| = -x$

{y=a&not(x<0)} y:=y+x {y=a+|x|}

(6) (y=a&not(x<0))➔(y+x=a+|x|)

(5) {y+x=a+|x|} y:=y+x {y=a+|x}}

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(4) {y=a&not(x<0)} y:=y+x {y=a+|x|}

(4) Reduces to (5) and (6) by
Precondition Strengthening

(5) Follows from assignment axiom

(6) Because not(x<0) ➔ |x| = x

If then else

(1) $\{y=a \& x<0\} y:=y-x \{y=a+|x|\}$

(4) $\{y=a \& not(x<0)\} y:=y+x \{y=a+|x|\}$

$$\frac{\{y=a \& x<0\} y:=y-x \{y=a+|x|\} \quad \{y=a \& not(x<0)\} y:=y+x \{y=a+|x|\}}{\{y=a\} \text{ if } x < 0 \text{ then } y:= y-x \text{ else } y:= y+x \ \{y=a+|x|\}}$$

By the if_then_else rule

While

- We need a rule to be able to make assertions about **while** loops.
  - Inference rule because we can only draw conclusions if we know something about the body
  - Let's start with:

$$\frac{\{\ ?\ \}\ C\ \{\ ?\ \}}{\{\ ?\ \}\ \textbf{while}\ B\ \textbf{do}\ C\ \textbf{od}\ \{\ P\ \}}$$

While

- The loop may never be executed, so if we want P to hold after, it had better hold before, so let's try:

$$\frac{\{\ ?\ \}\ C\ \{\ ?\ \}}{\{\ P\ \}\ \textbf{while}\ B\ \textbf{do}\ C\ \textbf{od}\ \{\ P\ \}}$$

While

- If all we know is P when we enter the **while** loop, then we all we know when we enter the body is (P and B)
- If we need to know P when we finish the **while** loop, we had better know it when we finish the loop body:

$$\frac{\{\,P \text{ and } B\}\ \ C\ \ \{\,P\,\}}{\{\,P\,\}\ \ \textbf{while}\ \ B\ \ \textbf{do}\ \ C\ \textbf{od}\ \ \{\,P\,\}}$$

While

- We can strengthen the previous rule because we also know that when the loop is finished, **not B** also holds
- Final **while** rule:

$$\frac{\{\ P \text{ and } B\ \}\ \ C\ \ \{\ P\ \}}{\{\ P\ \}\ \textbf{while}\ \ B\ \ \textbf{do}\ \ C\ \textbf{od}\ \ \{\ P \text{ and not } B\ \}}$$

While

$$\frac{\{\ P \text{ and } B\ \}\ \ C\ \ \{\ P\ \}}{\{\ P\ \}\ \textbf{while}\ \ B\ \ \textbf{do}\ \ C\ \textbf{od}\ \ \{\ P \text{ and not } B\ \}}$$

- P satisfying this rule is called a *loop invariant* because it must hold before and after the each iteration of the loop

While

- **While** rule generally needs to be used together with precondition strengthening and postcondition weakening
- There is NO algorithm for computing the correct P; it requires intuition and an understanding of why the program works

Example

- Let us prove

  {x>= 0 and x = a}

  fact := 1;

  while x > 0 do (fact := fact * x; x := x –1) od

  {fact = a!}

Example

- We need to find a condition P that is true both before and after the loop is executed, and such that

$$(P \text{ and not } x > 0) \Rightarrow (fact = a!)$$

Example

- First attempt:

{a! = fact * (x!)}

- Motivation:
- What we want to compute: **a!**
- What we have computed: **fact**

  which is the sequential product of **a** down through **(x + 1)**
- What we still need to compute: **x!**

Example

By post-condition weakening suffices to show

1. {x>=0 and x = a}

   fact := 1;

   while x > 0 do (fact := fact * x; x := x –1) od

   {a! = fact * (x!) and not (x > 0)}

and

2. {a! = fact * (x!) and not (x > 0) } ➔ {fact = a!}

Problem

2. {a! = fact * (x!) and not (x > 0)} → {fact = a!}

- Don't know this if x < 0
- Need to know that x = 0 when loop terminates
- Need a new loop invariant
- Try adding x >= 0
- Then will have x = 0 when loop is done

Example

Second try, combine the two:

P = {a! = fact * (x!) and x >=0}

Again, suffices to show

1. {x>=0 and x = a}

   fact := 1;

   while x > 0 do (fact := fact * x; x := x –1) od

   {P and not x > 0}

and

2. {P and not x > 0} ➔ {fact = a!}

Example

- For 2, we need

{a! = fact * (x!) and x >=0 and not (x > 0)} ➔ {fact = a!}

But {x >=0 and not (x > 0)} ➔ {x = 0} so

fact * (x!) = fact * (0!) = fact

Therefore

{a! = fact * (x!) and x >=0 and not (x > 0)} ➔ {fact = a!}

Example

- For 1, by the sequencing rule it suffices to show

3. {x>=0 and x = a}
   fact := 1
   {a! = fact * (x!) and x >=0 }

And

4. {a! = fact * (x!) and x >=0}
   while x > 0 do
   (fact := fact * x; x := x –1) od
   {a! = fact * (x!) and x >=0 and not (x > 0)}

Example

- Suffices to show that

  {a! = fact * (x!) and x >= 0}

  holds before the while loop is entered and that if

  {(a! = fact * (x!)) and x >= 0 and x > 0}

  holds before we execute the body of the loop, then

  {(a! = fact * (x!)) and x >= 0}

  holds after we execute the body

Example

By the assignment rule, we have

{a! = 1 * (x!) and x >= 0}
fact := 1
{a! = fact * (x!) and x >= 0}

Therefore, to show (3), by precondition strengthening, it suffices to show

(x>= 0 and x = a) ➔
(a! = 1 * (x!) and x >= 0)

Example

(x>= 0 and x = a) ➔
(a! = 1 * (x!) and x >= 0)

holds because x = a ➔ x! = a!

Have that {a! = fact * (x!) and x >= 0}
holds at the start of the while loop

Example

To show (4):

{a! = fact * (x!) and x >=0}
while x > 0 do
(fact := fact * x; x := x –1)
od
{a! = fact * (x!) and x >=0 and not (x > 0)}

we need to show that

{(a! = fact * (x!)) and x >= 0}

is a loop invariant

Example

We need to show:

{(a! = fact * (x!)) and x >= 0 and x > 0}

(fact = fact * x; x := x – 1)

{(a! = fact * (x!)) and x >= 0}

We will use assignment rule, sequencing rule and precondition strengthening

Example

By the assignment rule, we have

{(a! = fact * ((x-1)!)) and x – 1 >= 0}

x := x – 1

{(a! = fact * (x!)) and x >= 0}

By the sequencing rule, it suffices to show

{(a! = fact * (x!)) and x >= 0 and x > 0}

fact = fact * x

{(a! = fact * ((x-1)!)) and x – 1 >= 0}

Example

By the assignment rule, we have that

{(a! = (fact * x) * ((x-1)!)) and x – 1 >= 0}

fact = fact * x

{(a! = fact * ((x-1)!)) and x – 1 >= 0}

By Precondition strengthening, it suffices to show that

((a! = fact * (x!)) and x >= 0 and x > 0) ➔

((a! = (fact * x) * ((x-1)!)) and x – 1 >= 0)

Example

However

$$fact * x * (x - 1)! = fact * (x!)$$

and $(x > 0) \rightarrow x - 1 >= 0$

since x is an integer,so

$$\{(a! = fact * (x!)) \text{ and } x >= 0 \text{ and } x > 0\} \rightarrow$$

$$\{(a! = (fact * x) * ((x-1)!)) \text{ and } x - 1 >= 0\}$$

Example

Therefore, by precondition strengthening

{(a! = fact * (x!)) and x >= 0 and x > 0}

fact = fact * x

{(a! = fact * ((x-1)!)) and x – 1 >= 0}

This finishes the proof