Programming Languages and Compilers (CS 421)

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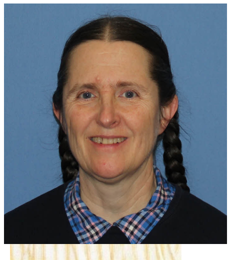

https://courses.engr.illinois.edu/cs421/sp2023

Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Two Problems

- Type checking
  - Question: Does exp. e have type τ in env Γ?
  - Answer: Yes / No
  - Method: Type derivation
- Typability
  - Question Does exp. e have some type in env. Γ? If so, what is it?
  - Answer: Type τ / error
  - Method: Type inference

Type Inference - Outline

- Begin by assigning a type variable as the type of the whole expression
- Decompose the expression into component expressions
- Use typing rules to generate constraints on components and whole
- Recursively find substitution that solves typing judgment of first subcomponent
- Apply substitution to next subcomponent and find substitution solving it; compose with first, etc.
- Apply comp of all substitution to orig. type var. to get answer

Type Inference - Example

- What type can we give to

  (fun x -> fun f -> f (f x))

- Start with a type variable and then look at the way the term is constructed

Type Inference - Example

- First approximate:

  $\{\ \}$ |- (fun x -> fun f -> f (f x)) : α

- Second approximate: use fun rule

$$\frac{\{x : \beta\} \mid\!- \text{(fun f -> f (f x))} : \gamma}{\{\ \} \mid\!- \text{(fun x -> fun f -> f (f x))} : \alpha}$$

- Remember constraint $\alpha \equiv (\beta \rightarrow \gamma)$

Type Inference - Example

- Third approximate: use fun rule

$$\frac{\dfrac{\{f : \delta\ ;\ x : \beta\}\ |\text{-}\ f\ (f\ x) : \varepsilon}{\{x : \beta\}\ |\text{-}\ (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \}\ |\text{-}\ (\text{fun } x \text{ -> fun } f \text{ -> } f(f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Fourth approximate: use app rule

$$\frac{\frac{\{f:\delta;\ x:\beta\} \mid\!- f : \varphi \to \varepsilon \quad \{f:\delta;\ x:\beta\} \mid\!- f\ x : \varphi}{\{f : \delta\ ;\ x : \beta\} \mid\!- (f\ (f\ x)) : \varepsilon}}{\frac{\{x : \beta\} \mid\!- (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}{\{\ \} \mid\!- (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Fifth approximate: use var rule, get constraint $\delta \equiv \varphi \rightarrow \varepsilon$, Solve with same
- Apply to next sub-proof

$$\dfrac{\dfrac{}{\{f:\delta;\ x:\beta\} \mid\!-\ f : \varphi \rightarrow \varepsilon} \quad \{f:\delta;\ x:\beta\} \mid\!-\ f\ x : \varphi}{\dfrac{\{f : \delta\ ;\ x : \beta\} \mid\!-\ (f\ (f\ x)) : \varepsilon}{\dfrac{\{x : \beta\} \mid\!-\ (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}{\{\ \} \mid\!-\ (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}}}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst: $\{\delta \equiv \varphi \to \varepsilon\}$

$$\frac{\dfrac{\dfrac{\dots \quad \{f:\varphi \to \varepsilon;\ x:\beta\} \mid\!- f\ x : \varphi}{\{f : \delta\ ;\ x : \beta\} \mid\!- (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \mid\!- (\text{fun } f \to f\ (f\ x)) : \gamma}}{\{\ \} \mid\!- (\text{fun } x \to \text{fun } f \to f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst: $\{\delta \equiv \varphi \to \varepsilon\}$ Use App Rule

$$\frac{\dfrac{\{f:\varphi\to\varepsilon;\ x:\beta\} \mid\!- f:\zeta\to\varphi \quad \{f:\varphi\to\varepsilon;\ x:\beta\} \mid\!- x:\zeta}{\ldots \quad \{f:\varphi \to \varepsilon;\ x:\beta\} \mid\!- f\ x : \varphi}}{\dfrac{\{f : \delta\ ;\ x : \beta\} \mid\!- (f\ (f\ x)) : \varepsilon}{\dfrac{\{x : \beta\} \mid\!- (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}{\{\ \} \mid\!- (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}}}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst: $\{\delta \equiv \varphi \to \varepsilon\}$
- Var rule: Solve $\zeta \to \varphi \equiv \varphi \to \varepsilon$ Unification

$$\dfrac{\dfrac{\dfrac{\{f:\varphi\to\varepsilon;\ x:\beta\} \mid\!- f:\zeta\to\varphi \quad \{f:\varphi\to\varepsilon;\ x:\beta\} \mid\!- x:\zeta}{\ldots \quad \{f:\varphi \to \varepsilon;\ x:\beta\} \mid\!- f\ x : \varphi}}{\{f : \delta\ ;\ x : \beta\} \mid\!- (f\ (f\ x)) : \varepsilon}}{\dfrac{\{x : \beta\} \mid\!- (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}{\{\ \} \mid\!- (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst: $\{\zeta \equiv \varepsilon, \varphi \equiv \varepsilon\} \circ \{\delta \equiv \varphi \rightarrow \varepsilon\}$
- Var rule: Solve $\zeta \rightarrow \varphi \equiv \varphi \rightarrow \varepsilon$ Unification

$$\dfrac{\dfrac{\dfrac{\dfrac{\{f:\varphi\rightarrow\varepsilon;\ x:\beta\} \mid\!- f:\zeta\rightarrow\varphi \quad \{f:\varphi\rightarrow\varepsilon;\ x:\beta\} \mid\!- x:\zeta}{\{f:\varphi \rightarrow \varepsilon;\ x:\beta\} \mid\!- f\ x : \varphi} \quad \ldots}{\{f : \delta\ ;\ x : \beta\} \mid\!- (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \mid\!- (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \} \mid\!- (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma); \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst: $\{\zeta \equiv \varepsilon,\ \varphi \equiv \varepsilon,\ \delta \equiv \varepsilon \rightarrow \varepsilon\}$
- Apply to next sub-proof

$$\dfrac{\dfrac{\dots \qquad \{f:\varepsilon\rightarrow\varepsilon;\ x:\beta\} \mid\!- \ x:\varepsilon}{\dots \quad \{f:\varphi \rightarrow \varepsilon;\ x:\beta\} \mid\!- \ f\ x : \varphi}}{\dfrac{\dfrac{\{f : \delta\ ;\ x : \beta\} \mid\!- \ (f\ (f\ x)) : \varepsilon}{\{x : \beta\} \mid\!- \ (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \} \mid\!- \ (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst: $\{\zeta \equiv \varepsilon, \varphi \equiv \varepsilon, \delta \equiv \varepsilon \to \varepsilon\}$
- Var rule: $\varepsilon \equiv \beta$

$$\dfrac{\dfrac{\dots \qquad \dfrac{}{\{f:\varepsilon \to \varepsilon;\ x:\beta\} \mid\!- \ x:\varepsilon}}{\dots \quad \{f:\varphi \to \varepsilon;\ x:\beta\} \mid\!- \ f\ x : \varphi}}{\dfrac{\dfrac{\{f : \delta\ ;\ x : \beta\} \mid\!- \ (f\ (f\ x)) : \varepsilon}{\{x : \beta\} \mid\!- \ (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \} \mid\!- \ (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst: $\{\varepsilon\equiv\beta\}o\{\zeta\equiv\varepsilon,\ \varphi\equiv\varepsilon,\ \delta\equiv\varepsilon\rightarrow\varepsilon\}$
- Solves subproof; return one layer

$$\dfrac{\dfrac{\dots \qquad \dfrac{}{\{f{:}\varepsilon\rightarrow\varepsilon;\ x{:}\beta\}\ |\!-\ x{:}\varepsilon}}{\dots \quad \{f{:}\varphi\rightarrow\varepsilon;\ x{:}\beta\}\ |\!-\ f\ x : \varphi}}{\dfrac{\dfrac{\{f : \delta\ ;\ x : \beta\}\ |\!-\ (f\ (f\ x)) : \varepsilon}{\{x : \beta\}\ |\!-\ (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \}\ |\!-\ (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst: $\{\varepsilon \equiv \beta, \zeta \equiv \beta, \varphi \equiv \beta, \delta \equiv \beta \rightarrow \beta\}$
- Solves this subproof; return one layer

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dots}{\dots \quad \{f:\varphi \rightarrow \varepsilon;\ x:\beta\} \mid\!- f\ x : \varphi}}{\{f : \delta ;\ x : \beta\} \mid\!- (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \mid\!- (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{}}{\{\ \} \mid\!- (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst: $\{\varepsilon \equiv \beta,\ \zeta \equiv \beta,\ \varphi \equiv \beta,\ \delta \equiv \beta \rightarrow \beta\}$
- Need to satisfy constraint $\gamma \equiv (\delta \rightarrow \varepsilon)$, given subst, becomes: $\gamma \equiv ((\beta \rightarrow \beta) \rightarrow \beta)$

$$\dfrac{\dfrac{\dfrac{\dots}{\{f : \delta\ ;\ x : \beta\}\ \vert\text{-}\ (f\ (f\ x)) : \varepsilon}}{\{x : \beta\}\ \vert\text{-}\ (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \}\ \vert\text{-}\ (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst:

$\{\gamma \equiv ((\beta\rightarrow\beta) \rightarrow \beta), \varepsilon\equiv\beta,\ \zeta\equiv\beta,\ \varphi\equiv\beta,\ \delta\equiv\beta\rightarrow\beta\}$

- Solves subproof; return one layer

$$\dfrac{\dfrac{\dfrac{\dots}{\{f : \delta\ ;\ x : \beta\}\ |\text{-}\ (f\ (f\ x)) : \varepsilon}}{\{x : \beta\}\ |\text{-}\ (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \}\ |\text{-}\ (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst:

$\{\gamma \equiv ((\beta \rightarrow \beta) \rightarrow \beta), \varepsilon \equiv \beta,\ \zeta \equiv \beta,\ \varphi \equiv \beta,\ \delta \equiv \beta \rightarrow \beta\}$

- Need to satisfy constraint $\alpha \equiv (\beta \rightarrow \gamma)$ given subst: $\alpha \equiv (\beta \rightarrow ((\beta \rightarrow \beta) \rightarrow \beta))$

$$\dfrac{\dfrac{\dots}{\{x : \beta\} \text{ |- (fun f -> f (f x))} : \gamma}}{\{\ \} \text{ |- (fun x -> fun f -> f (f x))} : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma)$;

Type Inference - Example

- Current subst:

$\{\alpha \equiv (\beta \rightarrow ((\beta\rightarrow\beta) \rightarrow \beta)),$

$\gamma \equiv ((\beta\rightarrow\beta) \rightarrow \beta), \varepsilon\equiv\beta, \zeta\equiv\beta, \varphi\equiv\beta, \delta\equiv\beta\rightarrow\beta\}$

- Solves subproof; return one layer

$$\frac{\overline{\{x : \beta\} \mid\!- (\text{fun f -> f (f x)}) : \gamma}}{\{\ \} \mid\!- (\text{fun x -> fun f -> f (f x)}) : \alpha}$$

Type Inference - Example

- Current subst:

$\{\alpha \equiv (\beta \rightarrow ((\beta \rightarrow \beta) \rightarrow \beta)),$

$\gamma \equiv ((\beta \rightarrow \beta) \rightarrow \beta), \varepsilon \equiv \beta,\ \zeta \equiv \beta,\ \varphi \equiv \beta,\ \delta \equiv \beta \rightarrow \beta\}$

- Done: $\alpha \equiv (\beta \rightarrow ((\beta \rightarrow \beta) \rightarrow \beta))$

{ } |- (fun x -> fun f -> f (f x)) : α

Type Inference Algorithm

Let infer $(\Gamma, e, \tau) = \sigma$

- Γ is a typing environment (giving polymorphic types to expression variables)
- e is an expression
- τ is a type (with type variables),
- σ is a substitution of types for type variables
- Idea: σ is substitution solving the constraints on type variables necessary for Γ |- $e : \tau$
- Should have $\sigma(\Gamma)$ |- $e : \sigma(\tau)$ valid

Type Inference Algorithm

infer (Γ, *exp*, τ) =

- Case *exp* of
  - Var *v* --> return Unify{$\tau \equiv$ freshInstance(Γ(*v*))}
    - Replace all quantified type vars by fresh ones
  - Const *c* --> return Unify{$\tau \equiv$ freshInstance φ } where Γ |- *c* : φ by the constant rules
  - fun *x* -> *e* -->
    - Let α, β be fresh variables
    - Let σ = infer ({x: α} + Γ, *e*, β)
    - Return Unify({$\sigma(\tau) \equiv \sigma(\alpha \rightarrow \beta)$}) $\circ\ \sigma$

Example of inference with Var Rule

Instance {'a -> 'w}   ('w a fresh variable)

{x: All 'a. ('a * 'b) list, y:All. 'b}|- x : (int * string) list

freshInstance(All 'a. ('a * 'b) list) = ('w * 'b) list

Unify {((int*string)list = ('w * 'b) list)} ={'w -> int, 'b -> string}

After substitution:

Instance {'a -> int}

{x:All 'a. ('a * string) list, y:All. string}|- x:(int * string) list

Type Inference Algorithm (cont)

- Case *exp* of
  - App (e_1 e_2) -->
    - Let α be a fresh variable
    - Let σ_1 = infer(Γ, e_1, $\alpha \rightarrow \tau$)
    - Let σ_2 = infer($\sigma_1(\Gamma)$, e_2, $\sigma_1(\alpha)$)
    - Return $\sigma_2 \circ \sigma_1$

Type Inference Algorithm (cont)

- Case *exp* of
  - If e_1 then e_2 else e_3 -->
    - Let σ_1 = infer(Γ, e_1, bool)
    - Let σ_2 = infer($\sigma_1\Gamma$, e_2, $\sigma_1(\tau)$)
    - Let σ_3 = infer($\sigma_2 \circ \sigma_1(\Gamma)$,$e_3$,$\sigma_2 \circ \sigma_1(\tau)$)
    - Return $\sigma_3 \circ \sigma_2 \circ \sigma_1$

Type Inference Algorithm (cont)

- Case *exp* of
  - let $x = e_1$ in e_2 -->
    - Let α be a fresh variable
    - Let σ_1 = infer(Γ, e_1, α)
    - Let σ_2 = infer({x:GEN($\sigma_1(\Gamma)$, $\sigma_1(\alpha)$)} + $\sigma_1(\Gamma)$, e_2, $\sigma_1(\tau)$)
    - Return $\sigma_2 \circ \sigma_1$

Type Inference Algorithm (cont)

- Case *exp* of
  - let rec $x = e_1$ in e_2 -->
    - Let α be a fresh variable
    - Let $\sigma_1 = \text{infer}(\{x: \alpha\} + \Gamma, e_1, \alpha)$
    - Let $\sigma_2 = \text{infer}(\{x:\text{GEN}(\sigma_1(\Gamma),\sigma_1(\alpha))\} + \sigma_1(\Gamma)\}, e_2, \sigma_1(\tau))$
    - Return $\sigma_2 \circ \sigma_1$

Type Inference Algorithm (cont)

- To infer a type, introduce type_of
- Let α be a fresh variable
- type_of (Γ, *e*) =
  - Let σ = infer (Γ, *e*, α)
  - Return σ (α)

- Need an algorithm for Unif