Programming Languages and Compilers (CS 421)

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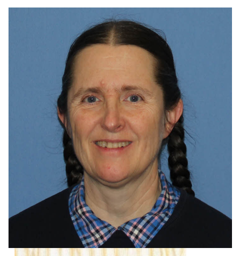

https://courses.engr.illinois.edu/cs421/sp2023

Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Format of Type Judgments

- A *type judgement* has the form

$$\Gamma \mid\!- \text{exp} : \tau$$

- Γ is a typing environment
  - Supplies the types of variables (and function names when function names are not variables)
  - Γ is a set of the form $\{ x : \sigma , \ldots \}$
  - For any x at most one σ such that $(x : \sigma \in \Gamma)$
- exp is a program expression
- τ is a type to be assigned to exp
- |- pronounced "turnstyle", or "entails" (or "satisfies" or, informally, "shows")

Axioms – Constants (Monomorphic)

$$\frac{}{\Gamma \mid\!- \; n : \text{int}}$$ (assuming *n* is an integer constant)

$$\frac{}{\Gamma \mid\!- \; \text{true} : \text{bool}} \qquad \frac{}{\Gamma \mid\!- \; \text{false} : \text{bool}}$$

- These rules are true with any typing environment
- Γ, *n* are meta-variables

Axioms – Variables (Monomorphic Rule)

Notation: Let $\Gamma(x) = \sigma$ if $x : \sigma \in \Gamma$

Note: if such σ exits, its unique

Variable axiom:

$$\frac{}{\Gamma \mid\!- \; x : \sigma} \quad \text{if } \Gamma(x) = \sigma$$

Simple Rules – Arithmetic (Mono)

Primitive Binary operators ($\oplus \in \{ +, -, *, \ldots\}$):

$$\frac{\Gamma \mid\!- e_1{:}\tau_1 \qquad \Gamma \mid\!- e_2{:}\tau_2 \qquad (\oplus){:}\tau_1 \rightarrow \tau_2 \rightarrow \tau_3}{\Gamma \mid\!- e_1 \oplus e_2 : \tau_3}$$

Special case: Relations ($\sim \in \{ <, >, =, <=, >= \}$):

$$\frac{\Gamma \mid\!- e_1 : \tau \quad \Gamma \mid\!- e_2 : \tau \quad (\sim){:}\tau \rightarrow \tau \rightarrow \text{bool}}{\Gamma \mid\!- e_1 \sim e_2 : \text{bool}}$$

For the moment, think τ is int

Example: {x:int} |- x + 2 = 3 :bool

What do we need to show first?

{x:int} |- x + 2 = 3 : bool

Example: {x:int} |- x + 2 = 3 :bool

What do we need for the left side?

$$\frac{\{x : int\} \mid\!- x + 2 : int \qquad\qquad \{x{:}int\} \mid\!- 3 : int}{\{x{:}int\} \mid\!- x + 2 = 3 : bool}\ \text{Bin}$$

Example: {x:int} |- x + 2 = 3 :bool

How to finish?

$$\dfrac{\dfrac{\{x:int\}\ |\text{-}\ x:int \quad \{x:int\}\ |\text{-}\ 2:int}{\{x : int\}\ |\text{-}\ x + 2 : int}\text{Bin} \qquad \{x:int\}\ |\text{-}\ 3 :int}{\{x:int\}\ |\text{-}\ x + 2 = 3 : bool}\text{Bin}$$

Example: {x:int} |- x + 2 = 3 :bool

Complete Proof (type derivation)

$$\dfrac{\dfrac{\dfrac{}{\{x:int\} \mid\!- \; x:int}\,\text{Var} \qquad \dfrac{}{\{x:int\} \mid\!- \; 2:int}\,\text{Const}}{\{x : int\} \mid\!- \; x + 2 : int}\,\text{Bin} \qquad \dfrac{}{\{x:int\} \mid\!- \; 3 :int}\,\text{Const}}{\{x:int\} \mid\!- \; x + 2 = 3 : bool}\,\text{Bin}$$

Simple Rules - Booleans

Connectives

$$\frac{\Gamma \ |\text{-}\ e_1 : \text{bool} \qquad \Gamma \ |\text{-}\ e_2 : \text{bool}}{\Gamma \ |\text{-}\ e_1 \ \&\& \ e_2 : \text{bool}}$$

$$\frac{\Gamma \ |\text{-}\ e_1 : \text{bool} \qquad \Gamma \ |\text{-}\ e_2 : \text{bool}}{\Gamma \ |\text{-}\ e_1 \ || \ e_2 : \text{bool}}$$

Type Variables in Rules

- If_then_else rule:

$$\frac{\Gamma \mid\!- \; e_1 : \text{bool} \quad \Gamma \mid\!- \; e_2 : \tau \quad \Gamma \mid\!- \; e_3 : \tau}{\Gamma \mid\!- \; (\text{if } e_1 \text{ then } e_2 \text{ else } e_3) : \tau}$$

- τ is a type variable (meta-variable)
- Can take any type at all
- All instances in a rule application must get same type
- Then branch, else branch and if_then_else must all have same type

Example derivation: if-then-else-

- Γ = {x:int, int_of_float:float -> int, y:float}

$$\frac{\Gamma \text{ |- (fun y -> y > 3) x : bool} \qquad \Gamma \text{ |- x+2 : int} \qquad \Gamma \text{ |- int\_of\_float y : int}}{\Gamma \text{ |- if (fun y -> y > 3) x then x + 2 else int\_of\_float y : int}}$$

Function Application

- Application rule:

$$\frac{\Gamma \;|\!\!-\; e_1 : \tau_1 \rightarrow \tau_2 \qquad \Gamma \;|\!\!-\; e_2 : \tau_1}{\Gamma \;|\!\!-\; (e_1\ e_2) : \tau_2}$$

- If you have a function expression e_1 of type $\tau_1 \rightarrow \tau_2$ applied to an argument e_2 of type τ_1, the resulting expression $e_1 e_2$ has type τ_2

Example: Application

- Γ = {x:int, int_of_float:float -> int, y:float}

$$\frac{\Gamma \text{ |- (fun y -> y > 3) : int -> bool} \qquad \Gamma \text{ |- x : int}}{\Gamma \text{ |- (fun y -> y > 3) x : bool}}$$

Fun Rule

- Rules describe types, but also how the environment Γ may change
- Can only do what rule allows!
- fun rule:

$$\frac{\{x : \tau_1\} + \Gamma \text{ |- } e : \tau_2}{\Gamma \text{ |- fun } x \text{ -> } e : \tau_1 \rightarrow \tau_2}$$

Fun Examples

$$\frac{\{y : int\} + \Gamma \text{ |- } y + 3 : int}{\Gamma \text{ |- } fun\ y \text{ -> } y + 3 : int \rightarrow int}$$

$$\frac{\{f : int \rightarrow bool\} + \Gamma \text{ |- } f\ \ 2 :: [true] : bool\ list}{\Gamma \text{ |- } (fun\ f \text{ -> } (f\ \ 2) :: [true]) : (int \rightarrow bool) \rightarrow bool\ list}$$

(Monomorphic) Let and Let Rec

- let rule:

$$\frac{\Gamma \mid\!- \; e_1 : \tau_1 \qquad \{x : \tau_1\} + \Gamma \mid\!- \; e_2 : \tau_2}{\Gamma \mid\!- \; (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

- let rec rule:

$$\frac{\{x : \tau_1\} + \Gamma \mid\!- \; e_1 : \tau_1 \quad \{x : \tau_1\} + \Gamma \mid\!- \; e_2 : \tau_2}{\Gamma \mid\!- \; (\text{let rec } x = e_1 \text{ in } e_2) : \tau_2}$$

Example

- Which rule do we apply?

$$\frac{?}{\{\} \mid\!- \text{(let rec one = 1 :: one in let x = 2 in fun y -> (x :: y :: one)) : int} \rightarrow \text{int list}}$$

Example

- Let rec rule:

② {one : int list} |- (let x = 2 in fun y -> (x :: y :: one)) : int → int list

① {one : int list} |- (1 :: one) : int list

$$\frac{\{one : int\ list\} \vdash (1 :: one) : int\ list \qquad \{one : int\ list\} \vdash (let\ x = 2\ in\ fun\ y \rightarrow (x :: y :: one)) : int \rightarrow int\ list}{\{\} \vdash (let\ rec\ one = 1 :: one\ in\ let\ x = 2\ in\ fun\ y \rightarrow (x :: y :: one)) : int \rightarrow int\ list}$$

{} |- (let rec one = 1 :: one in
let x = 2 in
fun y -> (x :: y :: one)) : int → int list

Proof of 1

- Which rule?

{one : int list} |- (1 :: one) : int list

Proof of 1

- Binary Operator

③ {one : int list} |- 1: int

④ {one : int list} |- one : int list

$$\frac{\{one : int\ list\} \vdash 1 : int \qquad \{one : int\ list\} \vdash one : int\ list}{\{one : int\ list\} \vdash (1 :: one) : int\ list}$$

where (::) : int → int list → int list

Proof of 1

③ Constant Rule

$$\frac{\text{Constant Rule}}{\{one : int\ list\} \mid\!- 1 : int}$$

④ Variable Rule

$$\frac{\text{Variable Rule}}{\{one : int\ list\} \mid\!- one : int\ list}$$

$$\frac{\{one : int\ list\} \mid\!- 1 : int \qquad \{one : int\ list\} \mid\!- one : int\ list}{\{one : int\ list\} \mid\!- (1 :: one) : int\ list}$$

750 minutes

Proof of 2

- Let Rule

$$\frac{\{one : int\ list\} \mid\!- 2:int \qquad \{x:int;\ one : int\ list\} \mid\!- fun\ y \to (x :: y :: one)) : int \rightarrow int\ list}{\{one : int\ list\} \mid\!- (let\ x = 2\ in\ fun\ y \to (x :: y :: one)) : int \rightarrow int\ list}$$

Proof of 2

- Constant

⑤ {x:int; one : int list} |- fun y -> (x :: y :: one)) : int → int list

{one : int list} |- 2:int

{one : int list} |- (let x = 2 in fun y -> (x :: y :: one)) : int → int list

Proof of 5

?

{x:int; one : int list} |- fun y -> (x :: y :: one))

: int $\rightarrow$ int list

Proof of 5

?

{y:int; x:int; one : int list} |- (x :: y :: one) : int list

{x:int; one : int list} |- fun y -> (x :: y :: one))

: int → int list

By the Fun Rule

Proof of 5

⑥ ⑦

$$\frac{\dfrac{\dfrac{?}{\{y:int;\ x:int;\ one:int\ list\}\ \vdash x:int} \qquad \dfrac{?}{\{y:int;\ x:int;\ one:int\ list\}\ \vdash (y :: one) : int\ list}}{\{y:int;\ x:int;\ one : int\ list\}\ \vdash (x :: y :: one) : int\ list}}{\{x:int;\ one : int\ list\}\ \vdash fun\ y \text{ -> } (x :: y :: one)) : int \rightarrow int\ list}$$

By BinOp where (::) : int → int list → int list

Proof of 6

⑥ Variable Rule

{y:int; x:int; one:int list} |- x:int

⑦ ?

{y:int; x:int; one:int list} |- (y :: one) : int list

{y:int; x:int; one : int list} |- (x :: y :: one) : int list

{x:int; one : int list} |- fun y -> (x :: y :: one)) : int $\rightarrow$ int list

Proof of 7

- Binary Operation Rule

$$\dfrac{\dfrac{?}{\{y:int;\ \ldots\} \mid\!- y:int} \quad \dfrac{\dfrac{?}{\{\ldots;\ one:int\ list;\ldots\}}}{\mid\!- one : int\ list}}{\{y:int;\ x:int;\ one : int\ list\} \mid\!- (y :: one) : int\ list}$$

By BinOp where (::) : int → int list → int list

Proof of 7

Variable Rule
{...; one:int list;...}
Variable Rule
{y:int; ...} |- y:int
|- one : int list
{y:int; x:int; one : int list}|- (y :: one) : int list

Curry - Howard Isomorphism

- Type Systems are logics; logics are type systems
- Types are propositions; propositions are types
- Terms are proofs; proofs are terms

- Function space arrow corresponds to implication; application corresponds to modus ponens

Curry - Howard Isomorphism

- Modus Ponens

$$\frac{A \Rightarrow B \quad A}{B}$$

- Application

$$\frac{\Gamma \mid\!- \; e_1 : \alpha \rightarrow \beta \quad \Gamma \mid\!- \; e_2 : \alpha}{\Gamma \mid\!- \; (e_1 \; e_2) : \beta}$$