Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Forward Recursion: Examples

```
# let rec double_up list =
    match list
    with [ ] -> [ ]
      | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>
```

Base Case Operator Recursive Call

```
# let rec poor_rev list =
  match list
  with [] -> []
    | (x::xs) -> let r = poor_rev xs in r @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

Base Case Operator Recursive Call

Recursing over lists

The Primitive Recursion Fairy

```
# let rec fold_right f list b =
  match list
  with [] -> b
  | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
   <fun>
# fold_right
    (fun s -> fun () -> print_string s)
    ["hi"; "there"]
    ();;
therehi- : unit = ()
```

Forward Recursion: Examples

```
# let rec double_up list =
    match list
    with [ ] -> [ ]
       | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>
```

Base Case Operator Recursive Call

```
# let double_up =
     fold_right (fun x -> fun r -> x :: x :: r) list [ ]
```

Operator Recursive result Base Case

```
# double_up ["a";"b"];;
- : string list = ["a"; "a"; "b"; "b"]
```

Folding Recursion : Length Example

```
# let rec length list = match list
  with [ ] -> 0   (* Nil case *)
  | a :: bs -> 1 + length bs;;  (* Cons case *)
val length : 'a list -> int = <fun>
# let length list =
fold_right (fun a -> fun r -> 1 + r) list 0;;
val length : 'a list -> int = <fun>
# length [5; 4; 3; 2];;
- : int = 4
```

Encoding Forward Recursion with Fold

```
# let rec multList_fr list =
```

ACT 2

- let rec multList_fr list =
  match list

```
with [] -> 1
| (x::xs) -> let r = (multList_fr ns) in
                    (x * r)
```

Folding Recursion

- multList folds to the right
- Same as:

```
# let multList list =
    List.fold_right
    (fun x -> fun p -> x * p)
    list 1;;
val multList : int list -> int = <fun>
# multList [2;4;6];;
- : int = 48
```

Extra Material

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 =

val append : 'a list -> 'a list -> 'a list = <fun>
```

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 = match list1 with

val append : 'a list -> 'a list -> 'a list = <fun>
```

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2
val append : 'a list -> 'a list -> 'a list = <fun>
```

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2 | x::xs ->
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2 | x::xs -> x :: append xs list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
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Base Case

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2 | x::xs -> x :: append xs list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case   Operation   Recursive Call

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2 | x::xs -> x :: append xs list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case

Operation

Recursive Call

```
# let append list1 list2 =
   fold_right (fun x -> fun y -> x :: y) list1 list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
```

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2 | x::xs -> x :: append xs list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case   Operation   Recursive Call

```
# let append list1 list2 =
   fold_right (fun x -> fun y -> x :: y) list1 list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
# append [1;2;3] [4;5;6];;
 - : int list = [1; 2; 3; 4; 5; 6]
```

Terminology

- Available: An operation that can be executed by the current expression
- The fastest way to be unavailable is to be guarded by an abstraction (anonymous function, lambda lifted).
  - if (h x) then f x else (x + g x)
  - if (h x) then (fun x -> f x) else (g (x + x))

Not available

Terminology

- Tail Position: A subexpression s of expressions e, which is available and such that if evaluated, will be taken as the value of e
  - if (x>3) then $\boxed{\text{x + 2}}$ else $\boxed{\text{x - 4}}$
  - let x = g 5 in $\boxed{\text{x + 4}}$
- Tail Call: A function call that occurs in tail position
  - if (h x) then $\boxed{\text{f x}}$ else $\boxed{\text{(x + g x)}}$

End of Extra Material

Terminology

- Available: A function call that can be executed by the current expression
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  - if (h x) then (fun x -> f x) else (g (x + x))

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  - let x = 5 in $\boxed{\text{x + 4}}$
- Tail Call: A function call that occurs in tail position
  - if (h x) then $\boxed{\text{f x}}$ else $\boxed{\text{(x + g x)}}$

Tail Recursion

- A recursive program is tail recursive if all recursive calls are tail calls
- Tail recursive programs may be optimized to be implemented as loops, thus removing the function call overhead for the recursive calls
- Tail recursion generally requires extra "accumulator" arguments to pass partial results
  - May require an auxiliary function

Tail Recursion - length

- How can we write length with tail recursion?

```
let length list =
   let rec length_aux list acc_length =
         match list
          with [ ] -> acc_length
            | (x::xs) ->
              length_aux xs (1 + acc_length)
    in length_aux list  0
```

Extra Material

Your turn: num_neg – tail recursive

```
# let num_neg list =
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =




  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] ->
      | (x :: xs) ->




  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->



  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
        num_neg_aux xs  ?



  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
         num_neg_aux xs
           (if x < 0 then 1 + curr_neg
            else curr_neg)
  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
        num_neg_aux xs
          (if x < 0 then 1 + curr_neg
           else curr_neg)
  in num_neg_aux list  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
         num_neg_aux xs
           (if x < 0 then 1 + curr_neg
            else curr_neg)
  in num_neg_aux list 0
```

End of Extra Material

Tail Recursion - length

- How can we write length with tail recursion?

```
let length list =
    let rec length_aux list acc_length =
        match list
         with [ ] -> acc_length
           | (x::xs) ->
             length_aux xs (1 + acc_length)
  in length_aux list 0
```

accumulated value

initial acc value

combing operation

Iterating over lists

```
# let rec fold_left f a list =
  match list
  with [] -> a
  | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
   <fun>
# fold_left
   (fun () -> (fun s -> print_string s))
   ()
   ["hi"; "there"];;
hithere- : unit = ()
```

length, fold_left

```
let length list =
  fold_left
   (fun acc -> fun x -> 1 + acc) // comb op
   0  // initial accumulator cell value
   list
```

Extra Material

Your turn: num_neg, fold_left

```
let num_neg list =
   fold_left
    ?  // comb op



    ?  // initial accumulator cell value
    ?
```

Your turn: num_neg, fold_left

```
let num_neg list =
  fold_left
   ?  // comb op



   0  // initial accumulator cell value
   ?
```

Your turn: num_neg, fold_left

```
let num_neg list =
   fold_left
    (fun curr_neg -> fun x ->
      if x < 0 then 1 + curr_neg else curr_neg)
        // comb op
    0  // initial accumulator cell value
    ?
```

Your turn: num_neg, fold_left

```
let num_neg list =
   fold_left
    (fun curr_neg -> fun x ->
      if x < 0 then 1 + curr_neg else curr_neg)
        // comb op
    0  // initial accumulator cell value
    list
```

End of Extra Material

350 minutes

Extra Material

poor_rev – forward recursive

```
# let rec poor_rev list =
    match list with [] -> []
      | (x :: xs) -> poor_rev xs @ [x]
```

Tail Recursion - Example

```
# let rec rev_aux list revlist =
  match list with [ ] -> revlist
  | x :: xs -> rev_aux xs (x::revlist);;
val rev_aux : 'a list -> 'a list -> 'a list = <fun>

# let rev list = rev_aux list [ ];;
val rev : 'a list -> 'a list = <fun>
```

- What is its running time?

Comparison

- poor_rev [1;2;3] =
- (poor_rev [2;3]) @ [1] =
- ((poor_rev [3]) @ [2]) @ [1] =
- (((poor_rev [ ]) @ [3]) @ [2]) @ [1] =
- (([ ] @ [3]) @ [2]) @ [1]) =
- ([3] @ [2]) @ [1] =
- (3:: ([ ] @ [2])) @ [1] =
- [3;2] @ [1] =
- 3 :: ([2] @ [1]) =
- 3 :: (2:: ([ ] @ [1])) = [3; 2; 1]

Comparison

- rev [1;2;3] =
- rev_aux [1;2;3] [ ] =
- rev_aux [2;3] [1] =
- rev_aux [3] [2;1] =
- rev_aux [ ] [3;2;1] = [3;2;1]

Folding - Tail Recursion

```
# let rev list =
        fold_left
          (fun l -> fun x -> x :: l)     //comb op
          []                  //accumulator cell
          list
```

End of Extra Material

Folding

```
# let rec fold_left f a list = match list
  with [] -> a | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
  <fun>
```

fold_left f a [x_1; x_2;…;x_n] = f(…(f (f a x_1) x_2)…)x_n

```
# let rec fold_right f list b = match list
  with [ ] -> b | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
  <fun>
```

fold_right f [x_1; x_2;…;x_n] b = f x_1(f x_2 (…(f x_n b)…))

Folding

- Can replace recursion by fold_right in any forward primitive recursive definition
  - Primitive recursive means here it only recurses on immediate subcomponents of recursive data structure
- Can replace recursion by fold_left in any tail primitive recursive definition

Extra Material

How long will it take?

- Remember the big-O notation from CS 225 and CS 374
- Question: given input of size *n*, how long to generate output?
- Express output time in terms of input size, omit constants and take biggest power

How long will it take?

Common big-O times:

- Constant time $O(1)$
  - input size doesn't matter
- Linear time $O(n)$
  - double input $\Rightarrow$ double time
- Quadratic time $O(n^2)$
  - double input $\Rightarrow$ quadruple time
- Exponential time $O(2^n)$
  - increment input $\Rightarrow$ double time

Linear Time

- Expect most list operations to take linear time $O(n)$
- Each step of the recursion can be done in constant time
- Each step makes only one recursive call
- List example: multList, append
- Integer example: factorial

Quadratic Time

- Each step of the recursion takes time proportional to input
- Each step of the recursion makes only one recursive call.
- List example:

```
# let rec poor_rev list = match list
  with [] -> []
    | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

Exponential running time

- Poor worst-case running times on input of any size
- Each step of recursion takes constant time
- Each recursion makes two recursive calls
- Easy to write naïve code that is exponential for functions that can be linear

Exponential running time

```
# let rec slow n =
    if n <= 1
    then 1
    else 1+slow (n-1) + slow(n-2);;
val slow : int -> int = <fun>
# List.map slow [1;2;3;4;5;6;7;8;9];;
- : int list = [1; 3; 5; 9; 15; 25; 41; 67;
  109]
```

Recall: Tail Recursion

- A recursive program is tail recursive if all recursive calls are tail calls
- Tail recursive programs may be optimized to be implemented as loops, thus removing the function call overhead for the recursive calls
- Tail recursion generally requires extra "accumulator" arguments to pass partial results
  - May require an auxiliary function

Terminology

- Available: A function call that can be executed by the current expression
- The fastest way to be unavailable is to be guarded by an abstraction (anonymous function, lambda lifted).
  - if (h x) then f x else (x + g x)
  - if (h x) then (fun x -> f x) else (g (x + x))

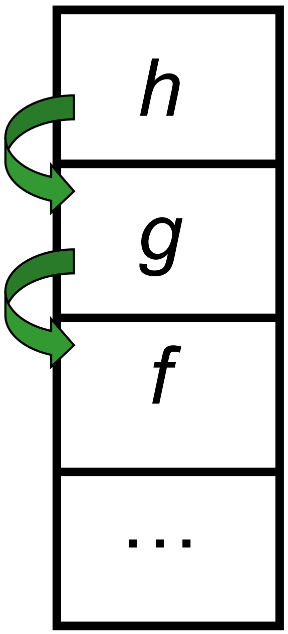
Not available

Terminology

- Tail Position: A subexpression s of expressions e, which is available and such that if evaluated, will be taken as the value of e
  - if (x>3) then x + 2 else x - 4
  - let x = 5 in x + 4
- Tail Call: A function call that occurs in tail position
  - if (h x) then f x else (x + g x)

An Important Optimization

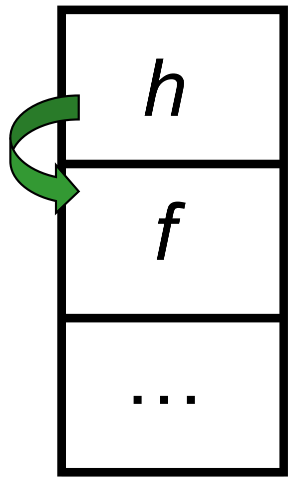
Normal call



- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?

An Important Optimization

Tail call



- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?
- Then *h* can return directly to *f* instead of *g*

End of Extra Material

Continuations

- A programming technique for all forms of "non-local" control flow:
  - non-local jumps
  - exceptions
  - general conversion of non-tail calls to tail calls
- Essentially it's a higher-order function version of GOTO

Continuations

- Idea: Use functions to represent the control flow of a program
- Method: Each procedure takes a function as an extra argument to which to pass its result; outer procedure "returns" no result
- Function receiving the result called a continuation
- Continuation acts as "accumulator" for work still to be done

Continuation Passing Style

- Writing procedures such that all procedure calls take a continuation to which to give (pass) the result, and return no result, is called continuation passing style (CPS)

Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.
- A formalization of non-local control flow in denotational semantics
- Possible intermediate state in compiling functional code

Why CPS?

- Makes order of evaluation explicitly clear
- Allocates variables (to become registers) for each step of computation
- Essentially converts functional programs into imperative ones
  - Major step for compiling to assembly or byte code
- Tail recursion (and forward recursion) easily identified

Other Uses for Continuations

- CPS designed to preserve order of evaluation
- Continuations used to express order of evaluation
- Can be used to change order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

Example

- Simple reporting continuation:

```
# let report x = (print_int x; print_newline( ) );;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```
# let addk (a, b) k = k (a + b);;
val addk : int * int -> (int -> ’a) -> ’a = <fun>
# addk (22, 20) report;;
2
- : unit = ()
```

Simple Functions Taking Continuations

- Given a primitive operation, can convert it to pass its result forward to a continuation
- Examples:

```
# let subk (x, y) k = k(x - y);;
val subk : int * int -> (int -> 'a) -> 'a = <fun>
# let eqk (x, y) k = k(x = y);;
val eqk : 'a * 'a -> (bool -> 'b) -> 'b = <fun>
# let timesk (x, y) k = k(x * y);;
val timesk : int * int -> (int -> 'a) -> 'a = <fun>
```

Nesting Continuations

```
# let add_triple (x, y, z) = (x + y) + z;;
val add_triple : int * int * int -> int = <fun>
# let add_triple (x,y,z)=let p = x + y in p + z;;
val add_triple : int * int * int -> int = <fun>
# let add_triple_k (x, y, z) k =
  addk (x, y) (fun p -> addk (p, z) k );;
val add_triple_k: int * int * int -> (int -> 'a) ->
  'a = <fun>
```

add_three: a different order

- # let add_triple (x, y, z) = x + (y + z);;
- How do we write add_triple_k to use a different order?

- let add_triple_k (x, y, z) k =

add_three: a different order

- # let add_triple (x, y, z) = x + (y + z);;
- How do we write add_triple_k to use a different order?

- let add_triple_k (x, y, z) k =

```
      addk (y,z) (fun r -> addk(x,r) k)
```